Neumann Problem in the Non-Classical Treatment for a Fourth Order Pseudoparabolic Equation

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***Abstract*— In the paper, the Neumann problem with non-classical boundary conditions not requiring agreement conditions in considered for a pseudoparabolic equation of fourth order. The equivalence of the conditions with classic boundary condition is substantiated in the case if the solution of the stated problem is sought in S.L.Sobolev space $W_p^{(2,2)}(G)$.**

***Keywords*— Neumann problem; pseudoparabolic equation; non-smooth coefficients equation**

Consider the equation

$$(V_{2,2}u)(x) \equiv D_1^2 D_2^2 u(x) + a_{2,1}(x) D_1^2 D_2 u(x) + \\ + a_{1,2}(x) D_1 D_2^2 u(x) + \sum_{\substack{i,j=0 \\ i+j<3}}^{2} a_{i,j}(x) D_1^i D_2^j u(x) = Z_{2,2}(x) \in L_p(G), \quad (1)$$

here $u(x) \equiv u(x_1, x_2)$ is a desired function determined on $G; a_{i,j}(x)$-are the given measurable functions on $G = G_1 \times G_2$, where $G_k = (0, h_k)$, $k = \overline{1,2}$; $Z_{2,2}(x)$ is a given measurable function on G; $D_k = \partial / \partial x_k$ is a generalized differentiation operator in S.L.Sobolev sense, D_k^0 is an identity operator.

Equation (1) is a hyperbolic equation that possesses two real characteristics $x_1 = const$, $x_2 = const$, the first and second of which are two-fold. Therefore, in some sense, equation (1) may be considered as a pseudoparabolic equation [1]. This equation is a generalization of Boussenesq – Liav equation [2] describing longitudinal waves in thin elastic bar with regard to lateral inertia effects.

It should be especially noted that first and second kind boundary value problems for hyperbolic equations were investigated in [3-6] and etc.

In the present paper, equation (1) is considered in the general case when the coefficients $a_{i,j}(x)$ are non-smooth functions that satisfy only the following conditions:

$$a_{i,j}(x) \in L_p(G), \; i = 0,1, \; j = 0,1;$$

$$a_{2,j}(x) \in L_{\infty,p}^{x_1,x_2}(G), \quad j = 0,1;$$

$$a_{i,2}(x) \in L_{p,\infty}^{x_1,x_2}(G), \; i = 0,1 \; .$$

Under these conditions, solution $u(x)$ of equation (1) will be sought in S.L.Sobolev space

$$W_p^{(2,2)}(G) \equiv \left\{ u(x) : D_1^i D_2^j u(x) \in L_p(G), \; i = \overline{0,2}, \; j = \overline{0,2} \right\},$$

where $1 \le p \le \infty$. For equation (1) we can give the classic form Neumann condition in the form (see. Fig.1)

$$\begin{cases} \left. \dfrac{\partial u(x_1,x_2)}{\partial x_1} \right|_{x_1=0} = \varphi_1(x_2); \; \left. \dfrac{\partial u(x_1,x_2)}{\partial x_2} \right|_{x_2=0} = \psi_1(x_1); \\ \left. \dfrac{\partial u(x_1,x_2)}{\partial x_1} \right|_{x_1=h_1} = \varphi_2(x_2); \; \left. \dfrac{\partial u(x_1,x_2)}{\partial x_2} \right|_{x_2=h_2} = \psi_2(x_1) \end{cases} \quad (2)$$

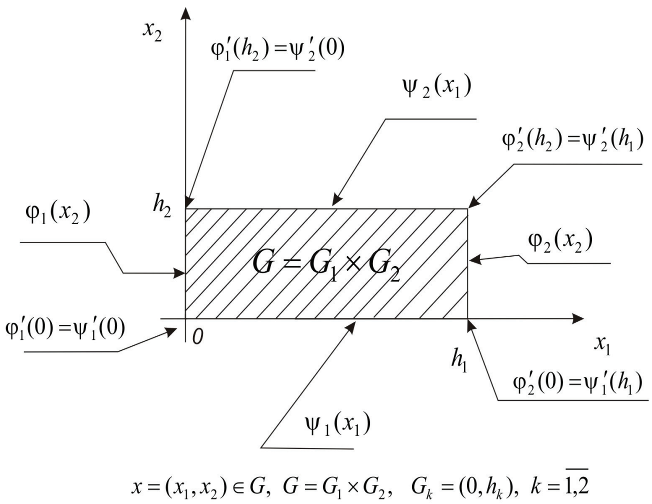


Figure. 1. Geometrical interpretation of Neumann classic conditions.

where $\varphi_k(x_2), \psi_k(x_1),\ k = 1,2$ are the given measurable functions on G. Obviously, in the case of conditions (2), in addition to the conditions

$$\varphi_k(x_2) \in W_p^{(2)}(G_2) \equiv \left\{\widetilde{\varphi}(x_2): D_2^j \widetilde{\varphi}(x_2) \in L_p(G_2),\ j = \overline{0,2}\right\} 1 \le p \le \infty;$$
$$\psi_k(x_1) \in W_p^{(2)}(G_1) \equiv \left\{\widetilde{\psi}(x_1): D_2^i \widetilde{\psi}(x_1) \in L_p(G_1),\ i = \overline{0,2}\right\} 1 \le p \le \infty$$

the given function should satisfy also the following agreement conditions

$$\begin{cases} \varphi_1'(h_2) = \psi_2'(0);\ \psi_1'(h_1) = \varphi_2'(0); \\ \varphi_1'(0) = \psi_1'(0); \varphi_2'(h_2) = \psi_2'(h_1). \end{cases} \tag{3}$$

Consider the following non - classic boundary conditions:

$$\begin{cases} V_{1,0}u \equiv D_1 u(0,0) = Z_{1,0} \in R; \\ V_{1,1}u \equiv D_1 D_2 u(0,0) = Z_{1,1} \in R; \\ (V_{1,2}u)(x_2) \equiv D_1 D_2^2 u(0,x_2) = Z_{1,2}(x_2) \in L_p(G_2); \\ V_{0,1}u \equiv D_2 u(0,0) = Z_{0,1} \in R; \\ (V_{2,1}u)(x_1) \equiv D_1^2 D_2 u(x_1,0) = Z_{2,1}(x_1) \in L_p(G_1); \\ V_{1,0}^{(h_1)}u \equiv D_1 u(h_1,0) = Z_{1,0}^{(h_1)} \in R; \\ V_{1,1}^{(h_1)}u \equiv D_1 D_2 u(h_1,0) = Z_{1,1}^{(h_1)} \in R; \\ (V_{1,2}^{(h_2)}u)(x_2) \equiv D_1 D_2^2 u(h_1,x_2) = Z_{1,2}^{(h_1)}(x_2) \in L_p(G_2); \\ V_{0,1}^{(h_2)}u \equiv D_2 u(0,h_2) = Z_{0,1}^{(h_2)} \in R; \\ V_{1,1}^{(h_2)} \equiv D_1 D_2 u(0\ h_2) = Z_{1,1}^{(h_2)} \in R; \\ (V_{2,1}^{(h_2)}u)(x_1) \equiv D_1^2 D_2 u(x_1,h_2) = Z_{2,1}^{(h_2)}(x_1) \in L_p(G_1) \end{cases} \tag{4}$$

If the function $u \in W_p^{(2,2)}(G)$ is a solution of the classic form Neumann problem (1), (2), then it is also a solution of problem (1), (4) for $Z_{i,j}$ and $Z_{i,j}^{(h_k)}$, determined by the following equalities:

$$Z_{1,0} = \varphi_1(0); Z_{1,1} = \varphi_1'(0) = \psi_1'(0); Z_{1,2}(x_2) = \varphi_1''(x_2);$$
$$Z_{0,1} = \psi_1(0); Z_{2,1}(x_1) = \psi_1''(x_1); Z_{1,0}^{(h_1)} = \varphi_2(0);$$
$$Z_{1,1}^{(h_1)} = \varphi_2'(0) = \psi_1'(h_1);\ Z_{1,2}^{(h_1)}(x_2) = \varphi_2''(x_2);$$
$$Z_{0,1}^{(h_2)} = \psi_2(0);\ Z_{1,1}^{(h_2)} = \psi_2'(0) = \varphi_1'(h_2); Z_{2,1}^{(h_2)}(x_1) = \psi_2''(x_1).$$

It is easy to prove the inverse one is also true. In other words, if the function $u \in W_p^{(2,2)}(G)$ is a solution of problem (1), (4) (see fig.2), then it is also a solution of problem (1), (2) for the following functions:

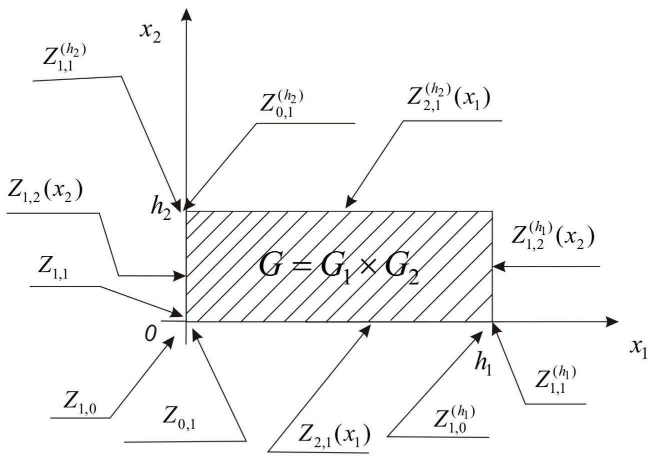
$$\varphi_1(x_2) = Z_{1,0} + x_2 Z_{1,1} + \int_0^{x_2} (x_2 - \tau) Z_{1,2}(\tau) d\tau; \tag{5}$$

$$\varphi_2(x_2) = Z_{1,0}^{(h_1)} + x_2 Z_{1,1}^{(h_1)} + \int_0^{x_2} (x_2 - \xi) Z_{1,2}^{(h_1)}(\xi) d\xi; \tag{6}$$

$$\psi_1(x_1) = Z_{0,1} + x_1 \cdot Z_{1,1} + \int_0^{x_1} (x - \eta) Z_{2,1}(\eta) d\eta; \tag{7}$$

$$\psi_2(x_1) = Z_{0,1}^{(h_2)} + x_1 \cdot Z_{1,1}^{(h_2)} + \int_0^{x_1} (x_1 - \mu) Z_{2,1}^{(h_2)}(\mu) d\mu. \tag{8}$$

Note that functions (5) - (8) possess an important property, more exactly, agreement conditions (3) for all $Z_{i,j}$, having the above-mentioned properties are fulfilled for them automatically. Therefore we can consider equalities (5)-(8) as a general form of the functions $\varphi_k(x_2), \psi_k(x_1) k = \overline{1,2}$, satisfying agreement conditions (3).



$$x = (x_1, x_2) \in G,\ \ G = G_1 \times G_2,\ \ G_k = (0, h_k),\ \ k = \overline{1,2}$$

Figure.2. Geometrical interpretation of Neumann conditions in non-classical treatment.

So, the classic form Neumann problem (1), (2) and in non-classic treatment (1) (4) (see. fig. 2) are equivalent in the general case. However, the Neumann problem in non-classical treatment (1), (4) is more natural by statement than problem (1), (2). This in connected with the fact that in the statement of problem (1), (4) the right sides of boundary conditions don't require additional conditions of agreement type. Note that some boundary value problems not requiring agreement conditions were substantiated in the author's papers [7-8].

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