Existence of the Solution of a Discrete Optimal Control Problem for Linear Concentrated Systems with Lions Special Quality Test

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***Abstract*— The paper deals with the existence of the solution of a discrete optimal control problem for linear concentrated systems with special quality test of Lions functional type.**

Keywords*— *optimal control; discrete problem; quality test; liner concentrated systems

I. INTRODUCTION

We investigate the convergence of the difference method for the solution of an optimal control problem for linear concentrated systems with a special quality test of Lions functional type. Note that a lot of papers have been devoted to the difference method for the solution of optimal control problems with concentrated systems. However, the problem under consideration considerably differs from the ones studied earlier. Therefore, study of convergence of the difference method in the present case both is of theoretical and practical interest.

II. STATEMENT AND DISCRETIZATION OF THE PROBLEM

Consider an optimal control problem on minimization of the functional

$$J(u) = \|x_0(\cdot, u) - x_T(\cdot, u)\|^2_{L_2^{(n)}(0,\,T)} \qquad (1)$$

on the set

$$U \equiv \left\{u = u(t) : u \in L_2^{(m)}(0, T),\ \|u\|_{L_2^{(m)}(0,\,T)} \le b_0\right\}$$

under the conditions:

$$x_p{}'(t) = A(t)x_p(t) + B(t)u(t) + f(t) \qquad (2)$$
$$p = 0, T,\ 0 < t < T,$$

$$x_0(0) = x_0,\quad x_T(T) = x_T\,, \qquad (3)$$

where $T > 0,\ b_0 > 0$ are the given numbers, $x_0,\ x_T$ are the given vectors, $A(t) = \{a_{ij}(t)\}$ are the matrices of order $n \times n$, $B(t) = \{b_{ik}(t)\}$ is a matrix of order $n \times m$, $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$ is a vector-column. We'll assume that $f \in L_2^{(n)}(0, T)$, $a_{ij} \in L_\infty(0, T)$, $i, j = \overline{1, n}$, $b_{ik} \in L_\infty(0, T)$, $i = \overline{1, n}$, $k = \overline{1, m}$.

At first, consider difference approximation of problem (1)-(3). To this end, partition the segment $[0, T]$ into N parts by the points $\left\{t_k,\ k = \overline{0, N}\right\}$:

$0 \le t_0 < t_1 < \ldots < t_{N-1} < t_N = T$, having accepted these points as nodes, we replace equation (2) by difference equations with the help of the Euler's simplest explicit diagram. As a result, we get the following discrete optimal control problem on minimization of the function

$$I_N([u]_N) = \sum_{k=0}^{N-1} \left|x_0{}^k - x_T{}^k\right|^2 \Delta t_k \qquad (4)$$

on the set

$$U_N = \{[u]_N : [u]_N = (u_0, u_1, \ldots, u_{N-1}),\ u_k \in R^m,$$
$$k = \overline{0, N-1},\ \left(\sum_{k=0}^{N-1} |u_k|_m^2 \Delta t_k\right)^{1/2} \le b_0\}$$

under the conditions

$$x_0{}^{k+1} = x_0{}^k + \Delta t_k\left(A_k x_0{}^k + B_k u_k + f_k\right),$$

$$k = \overline{0,\ N-1}, \qquad (5)$$

$$x_T^{\ k+1} = x_T^{\ k} + \Delta t_k \left(A_k x_T^{\ k} + B_k u_k + f_k \right),$$

$$k = \overline{N-1,\ 0}, \qquad (6)$$

$$x_0^{\ 0} = x_0, \quad x_T^{\ N} = x_T, \qquad (7)$$

where

$$\Delta t_k = t_{k+1} - t_k,\ A_k = \{a_{ij}^{\ k}\},\ B_k = \{b_{ir}^{\ k}\};$$

$$a_{ij}^k = \frac{1}{\Delta t_k} \int_{t_k}^{t_{k+1}} a_{ij}(t)dt, \quad b_{ir}^k = \frac{1}{\Delta t_k} \int_{t_k}^{t_{k+1}} b_{ir}(t)dt, \ ,$$

$$j = \overline{1,n}\ \ , r = \overline{1,n}$$

$$f_k = \left(f_{1k}, f_{2k}, \dots, f_{nk} \right),$$
$$f_{ik} = \frac{1}{\Delta t_k} \int_{t_k}^{t_{k+1}} f_i(t)dt,\ i = \overline{1,\ N},\ k = \overline{0,\ N-1}. \qquad (8)$$

$x_p^{\ k} = x_p^{\ k}([u]_N),\ p = 0, T,\ k = \overline{0,\ N}$ is the solution of difference diagram (5)-(7) for $[u]_N \in U_N$.

Introduce a space $L_{2N}^{(m)}$ of discrete control functions $[u]_N = (u_0, u_1, \dots, u_{N-1})$, $[v]_N = (v_0, v_1, \dots, v_{N-1})$ with the scalar product

$$< [u]_N, [v]_N >_{L_{2N}^{(m)}} = \sum_{k=0}^{N-1} \Delta t_k < u_k, v_k >_{R^m}$$

and with the norm

$$\|[u]_N\|_{L_{2N}^{(m)}} = \left(\sum_{k=0}^{N-1} \Delta t_k |u_k|_m^2 \right)^{1/2}.$$

The space $L_{2N}^{(m)}$ is a difference analogy of the space $L_2^{(m)}(0,\ T)$ corresponding to the partition $\left\{ t_k,\ k = \overline{0,\ N} \right\}$ of the segment $[0,\ T]$.

Thus, problem (1)-(3) considered in the space $L_2^{(m)}(0,\ T)$ for each entire $N \geq 1$ and partition $\left\{ t_k,\ k = \overline{0,\ N} \right\}$ of the segment $[0,\ T]$ corresponds to discrete optimal control problem (4)-(7) considered in the space $L_{2N}^{(m)}$.

For each $[u]_N \in U_N$ consider difference diagram (5)-(7).

Theorem 1. Let the elements of the matrices $A(t),\ B(t)$ be bounded and measurable functions on the segment $[0,\ T]$, $f \in L_2^{(n)}(0,\ T)$. Let furthermore, $d_N = \max\limits_{0 \leq k \leq N-1} \Delta t_k \leq \frac{T}{N} M$, $M = const > 0$.

Then the inequality holds for: $\forall [u]_N \in U_N$

$$\max_{0 \leq k \leq N} \left| x_p^{\ k}([u]_N) \right|_n \leq c_0, \quad p = 0, T, \qquad (9)$$

where $c_0 > 0$ is a constant independent of $k,\ N$.

Proof. Using (5) and (6) we have:

$$x_0^{\ k+1} = x_0 + \sum_{j=0}^{k} \Delta t_j \left(A_j x_0^{\ j} + B_j u_j + f_j \right),$$

$$k = \overline{0,\ N-1}, \qquad (10)$$

$$x_T^{\ k} = x_T + \sum_{j=k+1}^{N} \Delta t_{j-1} \left(A_{j-1} x_T^{\ j} + B_{j-1} u_{j-1} + f_{j-1} \right),$$

$$k = \overline{N-1,\ 0}. \qquad (11)$$

From equality (10) it is easy to get validity of the inequality

$$\left|x_0^{k+1}\right|_n \le \sqrt{n} A_{\max} d_N \sum_{j=0}^{k} \left|x_0^{j}\right|_n + \\ + \sqrt{n} B_{\max} \sum_{j=0}^{k} \left|u_j\right|_m \Delta t_j + \sum_{j=0}^{k} \left|f_j\right|_n \Delta t_j + \left|x_0\right|_n , \tag{12}$$

$$k = \overline{0,\, N-1},$$

where $A_{\max} = \underset{t\in[0,\,T]}{vrai \max} \left\| A(t) \right\|$,

$$B_{\max} = \underset{t\in[0,\,T]}{vrai \max} \left\| B(t) \right\|.$$

Using the Cauchy-Bunyakovsky inequality, from the last inequality we get:

$$\left|x_0^{k+1}\right|_n \le \sqrt{n} A_{\max} d_N \sum_{j=0}^{k} \left|x_0^{j}\right|_n + \sqrt{nT} B_{\max} \left\| [u]_N \right\|_{L_{2N}^{(m)}} + \\ + \sqrt{nT} \left\| [f]_N \right\|_{L_{2N}^{(m)}} + \left|x_0\right|_n ,$$

$$k = \overline{0,\, N-1}, \tag{13}$$

where $[f]_N = (f_0, f_1, \ldots, f_{N-1})$. Applying the discrete analogy of the Gronwall lemma [1, p.110], we get validity of the estimation:

$$\left|x_0^{k}\right|_n \le c_1 \left(\left|x_0\right|_n + \left\| [u]_N \right\|_{L_{2N}^{(m)}} + \left\| [f]_N \right\|_{L_{2N}^{(m)}} \right), \tag{14}$$

for $k = 0, 1, \ldots, N$, where $c_1 > 0$ is a constant independent of k.

Using inequality (11), similar to obtaining estimation (14), we can prove validity of the estimation:

$$\left|x_T^{k}\right|_n \le c_2 \left(\left|x_T\right|_n + \left\| [u]_N \right\|_{L_{2N}^{(m)}} + \left\| [f]_N \right\|_{L_{2N}^{(m)}} \right), \tag{15}$$

for $k = 0, 1, \ldots, N$, where $c_2 > 0$ is a constant independent of k.

Thus, from estimations (14), (15) we get validity of theorem's statement. Theorem 1 is proved.

III. Existence of the Solution of Discrete Optimal Control Problem

Now we'll study the existence of the solution of discrete problem (4)-(7).

Theorem 2. Let the conditions of theorem 1 fulfilled. Furthermore, the elements of the matrices A_k, B_k and vector-column f_k be defined by formulae (8). Then discrete optimal control problem (4)-(7) has at least one solution.

Proof. At first prove the continuity of function (4) on the set U_N. To this end take any control $[u]_N \in U_N$ and give it the increment $[h]_N$ such that $[u]_N + [h]_N \in U_N$.

Let $\lfloor x_p \rfloor_N = \lfloor x_p([u]_N) \rfloor_N$, $p = 0, T$ be the solution just of this system for $[u]_N + [h]_N \in U_N$. Then

$$\lfloor \Delta x_p \rfloor_N = \lfloor x_p([u]_N + [h]_N) \rfloor_N - \lfloor x_p([u]_N) \rfloor_N$$

will be the solution of the following system:

$$\Delta x_0^{k+1} = \Delta x_0^{k} + \Delta t_k \left(A_k \Delta x_0^{k} + B_k h_k \right),$$

$$k = \overline{0,\, N-1}, \tag{16}$$

$$\Delta x_T^{k+1} = \Delta x_T^{k} + \Delta t_k \left(A_k \Delta x_T^{k+1} + B_k h_k \right),$$

$$k = \overline{N-1,\, 0}, \tag{17}$$

$$\Delta x_0^{0} = 0, \quad \Delta x_T^{N} = 0. \tag{18}$$

We can write this system in the form

$$\Delta x_0^{k+1} = \sum_{j=0}^{k} \Delta t_j \left(A_j \Delta x_0^{j} + B_j h_j \right),$$

$$k = \overline{0,\, N-1}, \tag{19}$$

$$\Delta x_T^{k+1} = -\sum_{j=k+1}^{N} \Delta t_{j-1} \left(A_{j-1} \Delta x_T^{j} + B_{j-1} h_{j-1} \right),$$

$$k = \overline{0,\, N-1}, \tag{20}$$

From these equalities we can easily get validity of the inequalities:

$$\left|\Delta x_0^{\ k+1}\right|_n \le \sqrt{n}A_{\max}d_N\sum_{j=0}^{k}\left|\Delta x_0^{\ j}\right|_n + \\ +\sqrt{n}B_{\max}\sum_{j=0}^{k}\left|h_j\right|_m \Delta t_j,$$

$$k=\overline{0,\,N-1}, \tag{21}$$

$$\left|\Delta x_T^{\ k+1}\right|_n \le \sqrt{n}A_{\max}d_N\sum_{j=k+1}^{N}\left|\Delta x_T^{\ j}\right|_n + \\ +\sqrt{n}B_{\max}\sum_{j=k+1}^{N}\left|h_{j-1}\right|_m \Delta t_{j-1},$$

$$k=\overline{0,\,N-1}. \tag{22}$$

Applying in these inequalities the Cauchy-Bunyakovsky inequality and the discrete analogy of the Gronwall lemma, we get validity of the estimations

$$\left|\Delta x_p^{\ k}\right|_n \le c_3\left\|[h]_N\right\|_{L_{2N}^{(m)}},\ k=\overline{0,\,N},\ p=0,T, \tag{23}$$

where $c_8>0$ is a constant independent of k and $[h]_n$.

Now consider an increment of function (1.4) on the element $[u]_N \in U_N$. By means of the formula of this function we have:

$$\Delta I_N([u]_N) = I_N([u]_N + [h]_N) - I_N([u]_N) = \\ = 2\sum_{k=0}^{N-1} < x_0^{\ k} - x_T^{\ k},\, \Delta x_0^{\ k} - \Delta x_T^{\ k} >_{R^n} \Delta t_k + \\ + \sum_{k=0}^{N-1}\Delta t_k\left|\Delta x_0^{\ k}\right|_n^2 + \sum_{k=0}^{N-1}\Delta t_k\left|\Delta x_T^{\ k}\right|_n^2 - \\ -2\sum_{k=0}^{N-1} < \Delta x_0^{\ k},\, \Delta x_T^{\ k} >_{R^n} \Delta t_k \tag{24}$$

where $x_p^{\ k}$, $k=\overline{0,\,N}$, $p=0,T$ is the solution of system (1.5)-(1.7), for $[u]_N \in U_N$, and $\Delta x_p^{\ k}$, $k=\overline{0,\,N}$, $p=0,T$ is the solution of system (16)-(18).

By estimations (9), (23) and the Cauchy-Bunyakovsky inequality, from (24) we get validity of the inequality:

$$\left|\Delta I_N([u]_N)\right| \le c_4\left(\left\|[h]_N\right\|_{L_{2N}^{(m)}} + \left\|[h]_N\right\|_{L_{2N}^{(m)}}^2\right), \tag{25}$$

where $c_4>0$ is a constant independent of $[h]_n$. This inequality implies the continuity of the function $I_n([u]_n)$ on any element $[u]_N \in U_N$ i.e.

$$I_N([u]_N) \to 0 \ \text{ασ}\ \left\|[h]_N\right\|_{L_{2N}^{(m)}} \to 0. \tag{26}$$

By arbitrariness of the element $[u]_N \in U_N$ from (26) it follows the continuity of $I_n([u]_n)$ on the set U_N.

From the structure of the set U_N it is clear that it is a convex, closed and bounded set in a finite-dimensional space $L_{2N}^{(m)}$. It is known that such a set is compact in $L_{2N}^{(m)}$. By $I_n([u]_n)$ proved on this set, $I_n([u]_n) \ge 0$, $\forall [u]_N \in U_N$ is also continuous. Thus, all the conditions of the Weierstrass theorem are fulfilled [2,3]. By the same theorem, the function $I_n([u]_n)$ achieves its lower bound on the compact set U_N, i.e. the discrete optimal control problem (4)-(7) has at least one solution. Theorem 2 is proved.

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