Switching Discrete Control Problem

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***Abstract* - In this study, the authors investigate necessary optimality condition for switching optimal control problem. By using transformation, which described in this paper, the original switching discrete optimal control problem reduces to discrete optimal control problem and then by using Pontryagin maximum principle it is got necessary optimality condition for the original problem.**

Keywords -Optimal control, discrete system, maximum principle.

I. INTRODUCTION

A switching system is a particular kind of hybrid system that consists of several subsystems and switching law specifying the active subsystem at each time instants. Example of switching systems can be found in chemical processes, automotive systems, and electical circuit systems, etc. The available result in the literature on such problems can be specified into two categories,i.e, theoretical and practical [6,7] contain theoretical results. These results extension of the classical maximum principle or the dynamic programming approach to such problems. More complicated versions of the maximum principle under variations additional assumptions are proved in [6,8].

II. PROBLEM STATEMENT

In this study we consider the following discrete switching system:

Let
$t_0 \le t_0 + 1 \le \cdots \le t_1 \le t_1 + 1 \le \cdots t_{N-1} \le t_{N+1} \le \cdots \le t_N,$
be real numbers. Let us denote by T_k the discrete interval
$t_0 \le t_0 + 1 \le \cdots \le t_1 \le t_1 + 1 \le \cdots t_{N-1} \le t_{N+1} \le \cdots \le t_N.$

For any collection of functions

$x_k : T_k \to R^{n_k},\ k = 1,2,\ldots,N,$ define the vector of $p =$ (t_0,(t_1,x_1 (t_0),x_1 (t_1)), (t_2,x_2 (t_1), x_2 (t_2)),..,(t_1,x_(N-1) (t_(N-1)),x_N (t_N))) of dimension

$$d = 1 + N + 2\sum_{K=1}^{N} n_k.$$

On the discrete interval T_k consider the optimal control problem:

$$\begin{cases} x_k(t+1) = f_k\big(t, x_k(t), u_k(t)\big), \\ t \in \{t_{k-1}, t_{k-1}+1, \ldots, t_k\}, k = 1,2,\ldots,N, \\ n_j(p) = 0, j = 1,\ldots,q, \\ \varphi_i(p) \le 0 i = 1,\ldots,m, \\ J = \varphi_0(p) \to min. \end{cases} \quad \text{(A)}$$

Note1: *The time instants $t_0, t_1, t_2, \ldots, t_N$ are not fixed, a priori they just satisfy the above equality and inequality constraints on the vector p.*

Suppose the following assumptions are hold:

- every function f_k is defined and continuous on an open set;
- $Q_k \subset R^{1+n_k+r_k}$ and takes values in R^{n_k}; moreover, it has partial derivatives f_{kt}, f_{kx}, which are continuous on Q_k;
- the functions$\varphi_i(p)$ and$\mu_j(p)$ are defined an open set $P \subset R^d$ and continuously differentiable there;
- U_k are arbitrary sets in R^{rk}.

Definition 1.1 The tuple

$\omega = (t_0; t_k, x_k(t), u_k(t), k = 1,2,\ldots,N)$ is called admissible process in problem (A) if it satisfies all the constraints,and for every$k = 1,2, \dots, N$ there existis a compact set $\Omega_k \subset Q_k$ such that

$(t, x_k(t), u_k(t)) \in \Omega_k$ a.e onT_k.

***Definition** 1*An admissible process $\omega^0 = (\theta^0, x_k^0(t), u_k{}^0(t),)$ is called optimal in problem A if $J(\omega^0) \le J(\omega)$ for any admissible processω.

III. SOLUTION THE PROBLEM

To obtain optimality conditions in Problem A, we will reduce it to the following canonical autonomous optimal control problem of Pontryagin type on a fixed discrete interval $T = \{0,1,2, \dots, N\}$:

$$\begin{cases} x(t+1) = f(t, x(t), u(t), t \in T, \\ u \in U, (x, u) \in Q, \\ \mu_j(p) = 0, j = 1,2, \dots, q, \\ \varphi_i(p) \le 0, i = 1,2, \dots, m, \\ J = \varphi_0(p) \to min. \end{cases} \qquad (B)$$

Here $p(x(0), x(N)) \in R^{2n}$ is a vector of terminal values of the trajectory $x(t)$, and$Q$is an open set in R^{n+r}. Note that the problem B is special case of problem A,when the intermediate points apsent.

Suppose that problem B satisfies assumptions which is defined above. Then the following theorem holds.

Theorem *1(Pontryagin maximum principle for problem B)*. Let the set

$f(t, x(t), U(t)) = \{f(t, x(t), u(t) / u \in U\}$ is convex and process $\omega^0 = (x^0(t), u^0(t))$ gives Pontryagin minimum in problem B. Then there exist a collection $\lambda = (\alpha, \beta, c, \psi(.))$, where $\alpha=(\alpha_0, \alpha_1, \dots, \alpha_m)$ where all α_i are nonnegative (at least one of them positive).

$\beta = (\beta_0, \beta_1, \dots, \beta_q) \in R^q, c \in R^1$ and $\psi(.)$ is an dimensional function, which generates the Pontryagin functions$H(\psi, x, u) = \langle \psi, f(x, u) \rangle$ the terminal Lagrange function

$l(p) = \sum_{i=0}^{m} \alpha_i \varphi_i(p) + \sum_{j=0}^{q} \beta_i \eta_i(p)$, and satisfies the following conditions:

- Nnontriviality conditions : $(\alpha, \beta) \ne (0,0)$;
- Complementarity slackness conditions :

$$\alpha_i \varphi_i(p) = 0, i = 1,2, \dots, m;$$

- adjoint equations :

$$\psi(t-1) = -\psi(t) f_x(x^0(t), u^0(t), t \in T)$$

- transversality conditions :
- $\psi(0) = l(p^0)$,

$$\psi(N) = -l_{x(t)}(p^0)$$

- constancy of function H condition: $H(\psi(t), x^0(t), u^0(t)) = c, \forall t \in T$
- maximality condition: $max_{u \in U^0(t)} H(\psi(t), x^0(t), u) = c$

II Transformation

Let $(\theta, x(t), u(t))$ be an arbitrary admisssible process in problem A.

Introduce a new parameter

$\tau \in \{0,1,2, \dots, N\}$ and define functions:

$$p_k: \{\tau\} \to \{t_{k-1}, t_{k-1} + 1, \dots, t_k\},$$

$k = 1,2, \dots, N$ from the difference equations:

$$p_k(\tau + 1) = p_k(\tau) + z_k(\tau), p_k(k) = t_{k-1},$$

where z_k are discrete valued functions such that

$$\sum_{\tau=0}^{N-1} z_k(\tau) = t_k - t_{k-1}.$$

Define also functions

$y_k(\tau) = x_k(p_k(\tau))$and $v_k = u_k(p_k(\tau))$, $\tau \in \{0,1,2, \dots, N\}$,

$k = 0,1,2, \dots, N-1$.

They obviously satisfy the relations:

$$\begin{cases} y_k(\tau + 1) = y_k(\tau) + \end{cases}$$

$$\frac{\partial x_k}{\partial p_k} z_k(\tau+1), u_k \in U_k; \qquad (1)$$

Where $p_k(\tau+1) = p_k(\tau) + z_k(\tau)$,

$k = 0,1,2,\ldots,N-1$.

$$\begin{cases} p_2(0) - p_1(N) = 0, \\ p_3(0) - p_2(N) = 0, \\ p_N(0) - p_{N-1}(N) = 0, \end{cases} \qquad (2)$$

$$\begin{cases} \eta_j(\bar{p}) = 0, j = 1,2,\ldots,q, \\ \varphi_i(\hat{p}) \leq 0, i = 1,2,\ldots,m, \end{cases} \qquad (3)$$

For the simplicity, we use notation

$\hat{p} = (p, \widetilde{y}) = (p_1(0), (p_1(N), y_1(0), y_1(N)), (p_2(N), y_2(0), y_2(N)), \ldots, (p_N(1), y_N(0), y_N(N))$.

For the simplicity let us define vectors
$p = (p_1, p_2, \ldots, p_N)$, $y = (y_1, y_2, \ldots, y_N)$,
$v = (v_1, v_2, \ldots, v_N)$ and $z = (z_1, z_2, \ldots, z_N)$.

On the set of admissible processes
$\hat{\omega} = (p(\tau), y(\tau), v(\tau), z(\tau))$, satisfying above constraints (1)-(2) we will minimize the functional.
The obtained problem will be called Problem $\hat{B}$.Here, the state variables are p_k and y_k, the controls are v_k and

$$\breve{J}(\tilde{\omega}) = \varphi_0(\hat{p}) \rightarrow min$$

$z_k, k = 1,2,\ldots,N$, and the discrete interval{1,2,…,N} is not fixed. It is easy to see that problem$\hat{B}$ is the problem type A.

Note2: The following two correspondences can be established between the processes of Problems B and $\hat{B}$. As was shown above any admissible process $\omega = (\theta, x(t), u(t))$ of Problem B can be transformed to an admissible process
$\tilde{\omega} = (p(\tau), y(\tau), v(v), z(\tau))$
of problem $\tilde{B}$. Then by using maximum principle for the discrete optimal control problem in reference, we can obtain maximum principle for the problem A.

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