Numerical Solution to Optimal Control Problem for a Quasi-Nonlinear Parabolic Equation

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***Abstract*— The optimal control problem for quasi-nonlinear parabolic equation is considered in the work. The numerical algorithm of solution to the problem is offered, the questions of the computational realization are discussed and the instances of numerical experiments are given.**

***Keywords*— numerical solution; quasi-nonlinear parabolic equation; optimal control**

I. INTRODUCTION

The problem of increment of oil and gas recovery in ledges are stated a lot of problems, which solutions somehow or other are associated with optimal organization of the processes of actions on ledge and control by them.

Particularly, if we apply the thermal methods of stimulations then the necessity of determining the regimes and terms of thermo action on the ledge arises to provide maximum effect at minimal charges of thermal resources.

The theoretical investigations of this problem are associated with selecting of mathematical model of the process of thermo-mass transport in oil ledges and with statement of corresponding optimal control problems.

It is known that the thermalphysic character of the collector and ledge's fluids are the temperature functions which change its values slowly [1]. So the linear models are sufficiently similar to the process only at conditions considering little intervals of temperature variation [2]. Study the thermal processes at high intervals of temperature variation reduces to the quasi-linear heat equation. In connection with this high interest interprets development of the numerical solution algorithms of optimal control problems for such equations.

The optimal control problem for quasi-nonlinear parabolic equation is considered in the present work. The numerical algorithm of solution to the problem is offered, the questions of the computational realization are discussed and the instances of numerical experiments are given.

II. STATEMENT OF THE PROBLEM

The problem of determining the functions $v_1(t)$, $v_2(t)$, $u(x,t)$ satisfying the equation

$$C(u)\frac{\partial u}{\partial t}=\frac{1}{x}\frac{\partial}{\partial x}(x\lambda(u)\frac{\partial u}{\partial x})+\frac{q(t)}{x}\frac{\partial u}{\partial x}-\beta(t)u,$$

$$(x,t)\in G=\{x_c<x<x_R, 0<t\le T\}, \quad (1)$$

the initial and boundary conditions

$$u(x,0)=\varphi(x), \qquad x_c\le x\le x_R, \quad (2)$$

$$(\sigma_1 u-\sigma_2\lambda(u)\frac{\partial u}{\partial r})_{x=x_c}=v_1(t), \qquad 0\le t\le T, \quad (3)$$

$$(\sigma_3 u+\sigma_4\lambda(u)\frac{\partial u}{\partial r})_{x=x_R}=v_2(t), \ 0\le t\le T, \quad (4)$$

the constraints

$$v_{i\min}\le v_i(t)\le v_{i\max} \quad (i=1,2), \quad (5)$$

is considered when the functional

$$I(v_1,v_2)=\int_{x_c}^{x_R}\left(u(x,T)-u^*(x)\right)^2 x\,dx \quad (6)$$

takes on its minimal admissible value for given function $u^*(x)$. Here $v_{i\min}$, $v_{i\max}$ $(i=1,2)$ are the given numbers which characterized the limit admissible capability of the heat sources.

III. THE METHODS OF SOLUTION TO THE PROBLEM

The problem of minimization of the functional (6) at conditions (1)-(5) is solved by using two methods: conditional gradient (CG) and the projection of the gradient (PG), where it is required to find the functional gradient. The formulae for the gradient of the functional are deduced by the method of increments:

$$grad\ \ I=\begin{pmatrix}\sigma_1 x_c\lambda(u(x_c,t))\dfrac{\partial\phi(x_c,t)}{\partial x}+\sigma_2 x_c\phi(x_c,t),\\ \sigma_3 x_R\lambda(u(x_R,t))\dfrac{\partial\phi(x_R,t)}{\partial x}+\sigma_4 x_R y(x_R,t)\end{pmatrix} \quad (7)$$

here $\phi(x,t)$ - is the solution to the adjoint boundary problem:

$$-C(u)\frac{\partial\phi}{\partial t}=\frac{1}{x}\frac{\partial}{\partial x}(x\lambda(u)\frac{\partial\phi}{\partial x})-\frac{q(t)}{x}\frac{\partial\phi}{\partial x}-\beta(t)\phi$$

$$(x,t)\in G'=\{x_c<x<x_R, 0\le t<T\}$$

$$\phi(x,T) = -\frac{2(u(x,T) - u^*(x))}{C(u(x,T))}, \quad x_c \le x \le x_R \qquad (8)$$

$$\left(\sigma_1 \phi(x_c,t) + \sigma_2 (q(t)\phi(x,t) - \lambda(u)\frac{\partial \phi(x,t)}{\partial x})\right)_{x=x_c} = 0,$$

$$\left(\sigma_3 \phi(x_R,t) + \sigma_4 (\lambda(u)\frac{\partial \phi(x,t)}{\partial x} - q(t)\phi(x,t))\right)_{x=x_R} = 0,$$

$$0 \le t \le T.$$

So, the solution to the problem (1)-(6) is reduced to formulate the sequences $v_i^{(n)}(t)$ $(i=1,2)$ by using the method of conditional gradient or the projection of the gradient, at given initial admissible approaches $v_i^{(0)}(t)$. The step of the gradient method is selected from the condition of monotone decreasing of the functional (6) by the method of bisection [3].

We'll use the difference-iterational method on irregular grid at phase variables for numerical solution the problem, which is constructed on the base of aprior information about properties of the solution.

We'll use double-layer implicit difference schemes for boundary problems (1)-(4) and (8).
The value of the functional is calculated by the trapezoidal formula of the quadrature. The iteration process lasts till execution the condition:

$$I^{(n)} - I^{(n-1)} < \varepsilon \ , \ \varepsilon = 10^{-6}.$$

IV. The Results of Numerical Experiments

We'll introduce the examples of calculated computations, which carried out for the following values of the parameters in the problem (1)-(6):

$x_c = 0.01$, $x_R = 1$, $T = 0.12$, $v_{i\min} = 1$, $v_{i\max} = 7$ $(i=1,2)$

$\lambda(u) = u + 0.6$, $C(u) = u + 0.4$, $q(t) \equiv 1$,

$\beta(t) \equiv 0$, $\varphi(x) \equiv 0$,

$\sigma_1 = \sigma_3 = 1$, $\sigma_2 = \sigma_4 = 0$

The numerical experiment is carried out to solve the boundary-value problem (1)-(4) at given $v_i(t) \equiv v_i^*(t)$ $(i=1,2)$ and we'll use the function $u^*(x)$ as the solution to the problem at $t = T$. We'll assume $v_i(t) \equiv v_{i\min}$ $(i=1,2)$ as initial values.

The results of computations are given on fig 1-3 and on the table 1. The plots of model controls $v_i^*(t)$, $(i=1,2)$ and the plots of approximate optimal controls $v_i^{(14)}(t)$, $(i=1,2)$ which are obtained at 14-th iteration corresponding to MCG and MPG are given on the fig.1 and 2.

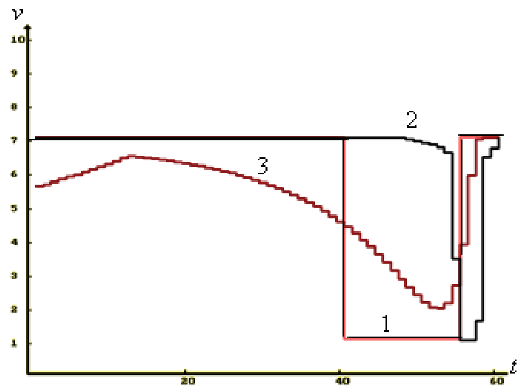

Figure.1. 1- $v_i^*(t)$ $(i=1,2)$, 2- $v_1^{(14)}(t)$, 3- $v_2^{(14)}(t)$ (CG)

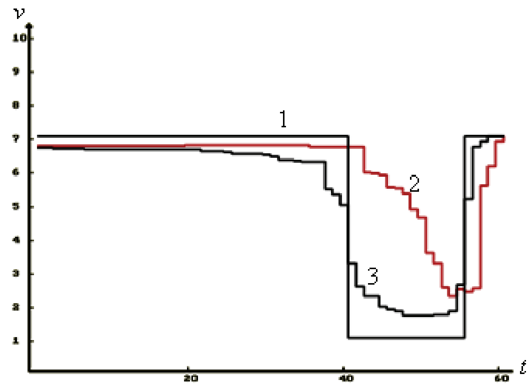

Figure. 2. 1- $v_i^*(t)$ $(i=1,2)$, 2- $v_1^{(14)}(t)$, 3- $v_2^{(14)}(t)$ (PG)

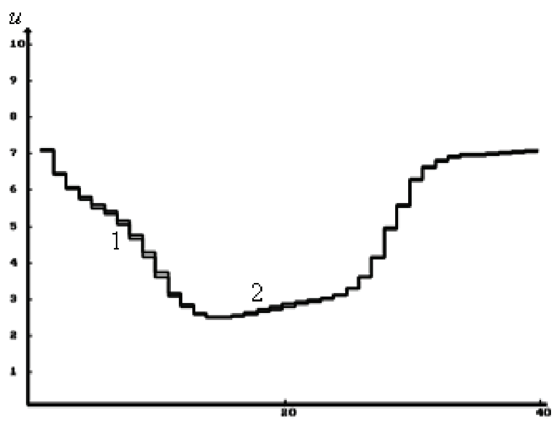

Figure. 3. 1- $u^*(x)$, 2- $u^{(14)}(x,T)$ (MCG и MPG)

The plots for the functions $u^*(x)$, for the solutions of the boundary-value problem (1)-(4) $u^{(14)}(x,T)$ at $t=T$ and for $v_i(t) \equiv v_i^{(14)}(t)$, $(i=1,2)$ are given on the fig.3.

As it seems from the figures 1-3, the approximate optimal controls are closed to exact controls upon the average, but the functions $u^*(x)$ and $u^{(14)}(x,T)$ are good coordinated in the metric of $C[x_c, x_R]$.

The changing of the functional values and the steps of the gradient methods at iterations are given on the table 1.

Let’s notice that the significant decreasing of the values of the functional is observed at first iterations for both methods. As it seems from the table 1 the MPG gives more exact results for functional than the MCG.

The computations are also carried out for other values of parameters of the problem. The obtained results aren’t differentiated from the results obtained in this numerical experiment. We can say from numerical results that if the optimal control has a “relay character” then it is preferentially to apply the method of condition gradient.

The numerical results have showed that the suggested algorithm gives sufficiently exact solutions to the considered problem. We can use this method to determine the optimal regimes of thermo action on the oil ledge.

TABLE 1.

n	$I^{(n)}$(MCG)	Step	$I^{(n)}$(MPG)	step
1	1.4739170	1.0000	1.4739170	1.0000
2	0.6361764	0.5000	0.6226563	0.1250
3	0.4603887	1.0000	0.6226563	1.0000
4	0.1689049	0.2500	0.0483407	0.5000
5	0.1417197	0.5000	0.0401408	0.2500
6	0.0350397	0.2500	0.0365495	0.5000
7	0.0342651	0.2500	0.0352101	0.1250
8	0.0179168	0.1250	0.0333435	1.0000
9	0.0146952	0.1250	0.0260074	0.0625
10	0.0130273	0.0625	0.0191121	1.0000
11	0.0114223	0.1250	0.0091035	0.0312
12	0.0096654	0.0625	0.0079553	0.5000
13	0.0078932	0.0625	0.0053461	0.1250
14	0.0077392	0.0156	0.0036101	0.0312

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