Necessary Optimality Conditions of Second Order in a Discrete Problem on Control of Fornazini-Markezini Systems

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***Abstract*— A stepwise discrete two-parametric boundary value problem of optimal control is considered. Necessary optimality conditions of first and second order are obtained.**

Keywords – Fornazini-Markenzini systems; necessary optimality conditions; control problem; admissible control; optimal control

I. Introduction

Various optimal control problems described by 2-D systems were studied in [1-3]. In the present paper, we consider a stepwise problem of optimal control of 2-D discrete systems. Under the assumption of openness of the control problem, necessary optimality conditions of first and second orders are obtained.

II. Statement of Problem

Consider a problem on minimum of the functional

$$S(u,v)=\varphi_1(c(x_1))+\varphi_2(d(x)) \tag{1}$$

under constraints

$$\begin{aligned} &u(x)\in U, \quad x=x_0,x_0+1,\dots,x_1-1,\\ &v(x)\in V, \quad x=x_1,x_1+1,\dots,X-1, \end{aligned} \tag{2}$$

$$\begin{gathered} z(t+1,x+1)=\\ =f(t,x,z(t,x),z(t+1,x),z(t,x+1)),\\ (t,x)\in D=\{(t,x):\ t=t_0,t_0+1,\dots,t_1;\\ x_0,x_0+1,\dots,\ x_1,x_1+1,\dots,X-1\}. \end{gathered} \tag{3}$$

$$\begin{gathered} z(t_0,x)=a(x), \quad x=x_0,x_0+1,\dots,\ x_1,x_1+1,\dots,X-1,\\ z(t,x_0)=b(t), \quad t=t_0,t_0+1,\dots,\ T,\\ a(x_0)=b(t_0), \end{gathered} \tag{4}$$

$$a(x)=\begin{cases} c(x), & x=x_0,x_0+1,\dots,x_1,\\ d(x), & x=x_1,x_1+1,\dots,X. \end{cases} \tag{5}$$

Here $f(t,x,z,\ell,m)$ is a given n-dimensional vector-function continuous in totality of variables together with partial derivatives with respect to (z,ℓ,m) to second order inclusively, t_0, T, x_0, x_1, X are the given numbers, moreover the difference $T-t_0$ and $X-x_0$ are natural numbers, $b(t)$ is a given n-dimensional discrete vector-function, $c(x)$ and $d(x)$ are n-dimensional vector-functions being the solutions of the following problems

$$\begin{gathered} c(x+1)=f_1(x,c(x),u(x)),\\ x=x_0,x_0+1,\dots,x_1-1,\\ c(x_0)=a(x_0), \end{gathered} \tag{6}$$

$$\begin{gathered} d(x+1)=f_2(x,d(x),v(x)),\\ x=x_1,x_1+1,\dots,X-1,\\ d(x_1)=G(c(x_1)), \end{gathered} \tag{7}$$

where $f_1(x,c,u)$, $f_2(x,d,v)$ are the given n-dimensional vector-functions continuous in totality of variables together with partial derivatives with respect to (c,u), (d,v), respectively, to second order inclusively, $U\subset R^r$, $V\subset R^r$ are the given non-empty, bounded and open sets, $G(c)$ is the twice continuously differentiable n-dimensional vector-function.

The pair $(u(x),v(x))$ with the above-mentioned properties is said do be an admissible control, and an admissible control delivering minimum to functional (1) under constraints (6)-(7) an optimal control.

III. Problem Solution (Necessary Optimality Conditions)

Assume that $(u^0(x),\ v^0(x),\ z^0(x),\ c^0(x),\ d^0(x))$ is a fixed admissible process.

Denote by $(\delta z(t,x),\ \delta c(x),\ \delta d(x))$ the solutions of the following problem:

$$\begin{gathered} \delta z(t+1,x+1)=f_z(t,x)\delta z(t,x)+\\ +f_\ell(t,x)\delta z(t+1,x)+f_m(t,x)\delta z(t,x+1)\\ \delta z(t_0,x)=\delta a(x), \qquad x=x_0,x_0+1,\dots,X,\\ \delta z(t,x_0)=0, \qquad t=t_0,t_0+1,\dots,T, \end{gathered}$$

$$\delta c(x+1)=\frac{\partial f_1(x)}{\partial c}\delta c(x)+\frac{\partial f_2(x)}{\partial u}\delta u(x),$$
$$x=x_0,x_0+1,\ldots,x_1-1,$$
$$\delta d(x+1)=\frac{\partial f_1(x)}{\partial d}\delta d(x)+\frac{\partial f_2(x)}{\partial v}\delta v(x),$$
$$x=x_1,x_1+1,\ldots,X,$$
$$\delta d(x_1)=\frac{\partial G(c(x_1))}{\partial c}\delta c(x_1).$$

Here and in sequel, $f_z(t,x)$, $f_\ell(t,x)$, $f_m(t,x)$, $\frac{\partial f_1(x)}{\partial c}$, $\frac{\partial f_2(x)}{\partial d}$ and etc. the denote the expressions of the form

$$f_z(t,x)\equiv f_z(t,x,z(t,x),z(t+1,x),z(t,x+1)),$$
$$f_\ell(t,x)\equiv f_\ell(t,x,z(t,x),z(t+1,x),z(t,x+1)),$$
$$f_m(t,x)\equiv f_m(t,x,z(t,x),z(t+1,x),z(t,x+1)),$$
$$\frac{\partial f_1(x)}{\partial c}\equiv\frac{f_1(t,x,z(t,x),z(t+1,x),z(t,x+1))}{\partial c},$$
$$\frac{\partial f_2(x)}{\partial d}\equiv\frac{f_2(t,x,z(t,x),z(t+1,x),z(t,x+1))}{\partial d}.$$

Assume

$$H(t,x,z,\ell,m,\psi)=\psi' f(t,x,z,\ell,m),$$
$$M_1(x,c,u,\rho_1)=\rho_1' f_1(x,c,u),$$
$$M_2(x,c,v,\rho_2)=\rho_2' f_2(x,d,v),$$

Where $(\psi(t,x),\ \rho_1(x),\ \rho_2(x))$ is a solution of the conjugated system.

Allowing for (6)-(7), using the traditional techniques, (see [1, 2]) we can show that the first and second variations of the quality functional have the following form

$$\delta^1 S(u,v;\ \delta u,\delta v)=$$
$$=-\sum_{x=x_0}^{x_1-1}\frac{\partial M_1'(x)}{\partial u}\delta u(x)-\sum_{x=x_0}^{X-1}\frac{\partial M_2'(x)}{\partial vu}\delta v(x);$$
$$\delta^2 S(u,v;\ \delta u,\delta v)=c'(x_1)\frac{\partial^2\ {}_1(c(x_1))}{\partial c^2}c(x_1)+$$
$$+d'(X)\frac{\partial^2\ {}_2(d(X))}{\partial d^2}d(X)-$$
$$-\sum_{t=t_0}^{T-1}\sum_{x=x_0}^{X-1}\Bigg[\delta z'(t,x)\frac{\partial^2 H(t,x)}{\partial z^2}\delta z(t,x)+$$
$$+\delta z'(t+1,x)\frac{\partial^2 H(t,x)}{\partial z^2}\delta z(t+1,x)+\delta z'(t+1,x)\times$$
$$\times\frac{\partial^2 H(t,x)}{\partial z\,\partial\ell}\delta z(t,x+1)+\delta z'(t,x+1)\frac{\partial^2 H(t,x)}{\partial\ell\,\partial z}\delta z(t,x)+$$
$$+\delta z'(t,x+1)\frac{\partial^2 H(t,x)}{\partial\ell^2}\delta z(t,x+1)\Bigg]-$$
$$-\sum_{x=x_0}^{x_1-1}\Bigg[\delta c'(x)\frac{\partial^2 M_1(x)}{\partial c^2}\delta c(x)+2\,\delta u'(x)\frac{\partial^2 M_1(x)}{\partial u\,\partial c}\delta c(x)+$$
$$+\delta u'(x)\frac{\partial^2 M_1(x)}{\partial u^2}\delta u(x)\Bigg]-$$
$$-\sum_{x=x_0}^{X-1}\Bigg[\delta d'(x)\frac{\partial^2 M_2(x)}{\partial d^2}\delta d(x)+$$
$$+2\,\delta v'(x)\frac{\partial^2 M_2(x)}{\partial v\,\partial d}\delta d(x)+\delta v'(x)\frac{\partial^2 M_2(x)}{\partial v^2}\delta v(x)\Bigg]$$

From the general theory it follows that along the optimal control $(u^0(x),\ v^0(x))$ for all $(\delta u(x),\delta v(x))$ the following relation is fulfilled:

$$\sum_{x=x_0}^{x_1-1}\frac{\partial M_1'(x)}{\partial u}\delta u(x)+\sum_{x=x_0}^{X-1}\frac{\partial M_2'(x)}{\partial v}\delta v(x)=0, \tag{8}$$

$$c'(x_1)\frac{\partial^2\varphi_1(c(x_1))}{\partial c^2}c(x_1)+d'(X)\frac{\partial^2\varphi_2(d(X))}{\partial d^2}d(X)-$$
$$-\sum_{t=t_0}^{T-1}\sum_{x=x_0}^{X-1}\Bigg[\delta z'(t,x)\frac{\partial^2 H(t,x)}{\partial z^2}\delta z(t,x)+$$
$$+\delta z'(t+1,x)\frac{\partial^2 H(t,x)}{\partial z^2}\delta z(t+1,x)+\delta z'(t+1,x)\times$$
$$\times\frac{\partial^2 H(t,x)}{\partial z\,\partial\ell}\delta z(t,x+1)+\delta z'(t,x+1)\frac{\partial^2 H(t,x)}{\partial\ell\,\partial z}\delta z(t,x)+$$
$$+\delta z'(t,x+1)\frac{\partial^2 H(t,x)}{\partial\ell^2}\delta z(t,x+1)\Bigg]-$$
$$-\sum_{x=x_0}^{x_1-1}\Bigg[\delta c'(x)\frac{\partial^2 M_1(x)}{\partial c^2}\delta c(x)+2\,\delta u'(x)\frac{\partial^2 M_1(x)}{\partial u\,\partial c}\delta c(x)+$$
$$+\delta u'(x)\frac{\partial^2 M_1(x)}{\partial u^2}\delta u(x)\Bigg]- \tag{9}$$

$$-\sum_{x=x_0}^{X-1}\left[\delta d'(x)\frac{\partial^2 M_2(x)}{\partial d^2}\delta d(x)+\right.$$
$$\left.2\,\delta v'(x)\frac{\partial^2 M_2(x)}{\partial v\,\partial d}\delta d(x)+\delta v'(x)\frac{\partial^2 M_2(x)}{\partial v^2}\delta v(x)\right]\geq 0.$$

Relations (8), (9) are implicit necessary optimality conditions of first and second orders, respectively.

Using them, we'll get necessary optimality conditions of first and second orders directly expressed by the parameters of problem (1)-(7).

By independence of variations $\delta u(x)$, $\delta v(x)$ of control functions $u^0(x)$ and $v^0(x)$ from identity (8) it follows

Theorem 1. For optimality of the admissible control $(u^0(x),\, v^0(x))$ it is necessary that the relation

$$\frac{\partial M_1(x)}{\partial u}=0, \text{ for all } x=x_0,x_0+1,\ldots,x_1-1,$$
$$\frac{\partial M_2(x)}{\partial v}=0, \text{ for all } x=x_1,x_1+1,\ldots,X-1. \quad (10)$$

be fulfilled.

Relations (10) is the system of Euler equations for the considered problem.

Definition 1. Each solution $(u^0(x),\, v^0(x))$ of the system of Euler equations is called a classical extremum.

It is clear that an optimal control is among the classic extremums. But their number may be rather great. Therefore, using inequality (9), it should be obtained necessary optimality conditions of second order directly expressed by the parameters of problem (1)-(7).

IV. Conclusion

A problem on optimal control of discrete two-parameter systems is considered. Necessary optimality conditions in the case of openness of the control domain, are obtained.

References

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