Optimal Control Problem for Impulsive Systems with Integral Boundary Conditions

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Abstract— **In this paper the optimal control problem is considered, when the state of the system is described by the impulsive differential equations with integral boundary conditions. By the help of the Banach contraction principle the existence and uniqueness of solution is proved for the corresponding boundary problem by the fixed admissible control. The first variation of the functional is calculated. Various necessary conditions of optimality of the first order are obtained by the help of the variation of the controls.**

Keywords*— *integral boundary conditions; singular control; optimal control problem; existence and uniqueness of the solution

I. INTRODUCTION

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences. There is a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs [1-3] and the references therein.

Many of the physical systems can be described better by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral conditions constitute a very interesting and important class of problems. They include two, three, multi and nonlocal boundary value problems as special cases, (see [4-6]). For boundary value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to the papers [8-10] and the references therein.

The optimal control problems with boundary conditions have been investigated by the authors [7].

In the present paper, we investigate an optimal control problem in which the state of the system is described by the differential equations with integral boundary conditions. Note that this problem is a natural generalization of the Cauchy problem. The matters of existence and uniqueness of solutions of the boundary value problem are investigated, first variations of the functional are calculated.

II. PROBLEM STATEMENT

Consider the following impulsive system of differential equations with integral boundary condition

$$\frac{dx}{dt} = f(t, x, u(t)), \qquad 0 \le t \le T, \tag{1}$$

$$x(0) + \int_0^T m(t)x(t)dt = C, \tag{2}$$

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), \quad i = 1,2,\dots,p, \qquad 0 < t_1 < t_2 < \dots < t_p < T, \tag{3}$$

$$u(t) \in U, \ t \in [0,T], \tag{4}$$

where $x(t) \in R^n$; $f(t,x,u)$ and $I_i(x)$, $i = 1,2,\dots,p$ are n-dimensional continuous functions and have the second order derivative with respect to (x,u); $C \in R^n$ is a given constant vector and $m(t)$ is an $n \times n$ matrix function; u is a control parameter; $U \in R^r$ is an open set.

It is required to minimize the functional

$$J(u) = \varphi(x(0), x(T)) + \int_0^T F(t,x,u)dt \tag{5}$$

on the solutions of boundary value problem (1)-(4).

Here, it is assumed, that the scalar functions $\varphi(x,y)$ and $F(t,x,u)$ are continuous by their own arguments and have continuous and bounded partial derivatives with respect to x, y and u up to second order, inclusively. Under the solution of the boundary value problem (1)-(3) corresponding to the fixed control parameter $u(\cdot) \in U$ we mean the function $x(t):[0,T] \to R^n$ that is absolutely continuous on $[0,T]$, $t \ne t_i$, $i = 1,2,\dots,p$ and continuous from the left for $t = t_i$, for which there exists a finite right limit $x(t_i^+)$ for $i = 1,2,\dots,p$. Denote the space of such functions by $PC([0,T], R^n)$. It is obvious that such a space is Banach with the norm $\|x\|_{PC} = vrai\max_{t\in[0,T]}|x(t)|$, where $|\cdot|$ is the norm in space R^n.

The admissible process $\{u(t), x(t,u)\}$ being the solution of problem (1)-(5), i.e. delivering minimum to functional (5) under restrictions (1)-(4), is said to be an optimal process, $u(t)$ an optimal control.

III. Existence of Solutions of Boundary Value Problem (1)-(3).

Introduce the following conditions:

1) Let $\|B\| < 1,$ where $B = \int_0^T m(t)dt$.

2) $f : [0,T]\times R^n \times R^r \to R^n$ and $I_i : R^n \to R^n$, $i = 1,2,..., p$ are continuous functions and there exist the constants $K \ge 0$, $l_i > 0$ $i = 1,2,..., p$, such that

$$|f(t,x,u) - f(t,y,u)| \le K|x - y|,$$

$$t \in [0,T], \quad x, y \in R^n, u \in R^r,$$

$$|I_i(x) - I_i(y)| \le l_i|x - y|, \quad x, y \in R^n$$

3) $$L = (1 - \|B\|)^{-1}\left[KTN + \sum_{i=1}^{p} l_i\right] < 1,$$

where $N = \max_{0\le t,s\le T}\|N(t,s)\|$,

$$N(t,s) = \begin{cases} E + \int_0^s m(\tau)d\tau, 0 \le t \le s \\ -\int_s^T m(\tau)d\tau, s \le t \le T \end{cases}.$$

Note that under condition 1) the matrix $E + B$ is invertible and the estimation $\|(E+B)^{-1}\| < (1-\|B\|)^{-1}$ holds.

Theorem 1. Let condition 1) is satisfied. Then the function $x(\cdot) \in AC([0,T], R^n)$ is an absolutely continuous solution of the boundary value problem (1)-(3) iff

$$x(t) = (E+B)^{-1}C + \int_0^T K(t,\tau)f(\tau, x(\tau), u(\tau))d\tau + \sum_{0<t_k<T} Q(t,t_k)I_k(x(t_k)),$$

where

$$K(t,\tau) = (E+B)^{-1}N(t,\tau),$$

$$Q(t,t_k) = \begin{cases} (E+B)^{-1}, 0 \le \tau_k \le t \\ -(E+B)^{-1}B, t \le \tau_k \le T \end{cases}.$$

Theorem 2. Let conditions 1)-3) be fulfilled. Then for any $C \in R^n$ and for each fixed admissible control, boundary value problem (1)-(3) has the unique solution that satisfies the following integral equation

$$x(t) = (E+B)^{-1}C + \int_0^T K(t,\tau)f(\tau, x(\tau), u(\tau))d\tau + \sum_{0<t_k<T} Q(t,t_k)I_k(x(t_k)).$$

IV. The Functional Increment Formula

Let

$$\{u, x = x(t,u)\}$$

and

$$\{\tilde{u} = u + \Delta u,\ \tilde{x} = x + \Delta x = x(t,\tilde{u})\}$$

be two admissible processes. After some sufficiently standard operations usually used in deriving optimality conditions of the first and second orders, for the increment of the functional we obtain the formula

$$\Delta J(u) = -\int_0^T \left\langle \frac{\partial H(t,\psi,x,u)}{\partial u}, \Delta u(t) \right\rangle dt + o(\|\Delta u\|)$$

where

$$H(t,\psi,x,u) = \langle \psi, f(t,x,u)\rangle - F(t,x,u),$$

vector function $\psi(t) \in R^n$ and vector $\lambda \in R^n$ is solution of the following adjoint problem (the stationary condition of the Lagrangian function by state)

$$\dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x} - m'(t)\lambda, \quad t \in [0,T],$$

$$\psi(t_i^+) - \psi(t_i) = -I'_{ix}(x(t_i))(I'_{ix}(x(t_i)) + E)^{-1}\psi(t_i),$$

$$i = 1,2,..., p,$$

$$\frac{\partial \varphi}{\partial x(0)} - \psi(0) + \lambda = 0, \quad \frac{\partial \varphi}{\partial x(T)} + \psi(T) = 0.$$

References

[1] M.Benchohra, J.Henderson, S.K.Ntouyas. Impulsive differential equations and inclusions. Hindawi Publishing Corparation, Vol. 2, New York, 2006.

[2] A.M.Samoilenko, N.A.Perestuk, Impulsive differential equations. Worlds Scientific, Singapore, 1995.

[3] N.A.Perestyk, V.A.Plotnikov, A.M.Samoilenko, N.V.Skripnik Differential Equations with impulse Effect: Multivalued Right-hand Sides with Discontinuities, DeGruyter Studies in Mathematics 40, Walter de Gruter Co, Berlin, 2011.

[4] A.Bouncherif, Second order boundary value problems with integral boundary conditions, Nonlinear Analysis, 70, 1 (2009), pp. 368-379.

[5] R.A.Khan, Existence and approximation of solutions of nonlinear problems with integral boundary conditions, Dynamic Systems and Applications, 14, (2005), pp. 281-296.

[6] A.Belarbi, M.Benchohra, A.Quahab, Multiple positive solutions for nonlinear boundary value problems with integral boundary conditions, Archivum Mathematicum, vol. 44, no. 1, pp. 1-7, 2008.

[7] M.F.Mekhtiyev, Sh.I.Djabrailov, Y.A.Sharifov, Necessary optimality conditions of second order in classical sense in optimal control problems of three-point conditions, J. of Automation and Information Sciences, 42(3), 2010, pp. 47-57.