Numerical Solution to an Inverse Problem for Quasilinear Parabolic Equation

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***Abstract*— In the work, we consider a process described by a quasilinear parabolic differential equation under some initial and boundary conditions. We set a problem of identifying the heat conductivity coefficient as a function of temperature value. Problems in such a statement arise when studying qualitative properties of materials and media which are not yielding to direct measurements, but studied by means of indirect measurements. To solve the problem formulated, we propose to seek the unknown function (heat conductivity coefficient) on the class of piecewise constant functions. With this end in view, we quantize the set of the phase state's values (temperature) by means of predetermined values. Thus the problem of finding the unknown function is reduced to a problem of determining the finite-dimensional vector. To solve the latter problem, we propose to use first-order numerical minimization methods. With this end in view, we derive formulas for the gradient of the target functional in the space of optimizable parameters.**

Keywords*— coefficient inverse problem; quasilinear parabolic equation; gradient of functional; adjoint problem; finite-dimensional optimization

I. INTRODUCTION

As is known, for high-temperature processes proceeding, for example, in plasma, the heat conductivity coefficient is a nonlinear function of temperature (and of density). In a series of problems, it is also a function of temperature gradient. Next, heat sources (right-hand members in the heat conduction equation) may depend on temperature if, for example, heat is released as a result of some chemical reaction. Heat capacity of a medium may depend on temperature as well.

Thus we come to a nonlinear heat conduction equation

$$\frac{\partial v(x,t,u)}{\partial t} = -\frac{\partial w}{\partial x} + f(x,t,u)$$

where the heat flow $w = w\left(x,t,u,\frac{\partial u}{\partial x}\right)$ is a nonlinear function of temperature $u(x,t)$ and of its derivative. If the heat flow is linearly dependent on the derivative $\frac{\partial u}{\partial x}$ and the Fourier law $w = -a(x,t,u)\frac{\partial u}{\partial x}$ is fulfilled, then we come to a quasilinear heat conduction equation

$$c(x,t,u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,t,u)\frac{\partial u}{\partial x}\right) + f(x,t,u),$$

$$c(x,t,u) > 0,\ a(x,t,u) > 0.$$

In this case the heat capacity c, heat conductivity coefficient a, and the right-hand member f (density of heat sources) depend on the temperature $u(x,t)$. In an inhomogeneous medium c, a, and f may be discontinuous functions of x and t (for different substances the dependency of c, a, and f on the temperature u may be distinct).

The case when the functions $c = c(u)$, $a = a(u)$, and $f = f(u)$ depend solely on temperature is typical:

$$c(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right) + f(u).$$

II. PROBLEM STATEMENT

In the work, we investigate a process described by a simplest quasilinear parabolic differential equation [1, 2]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(u)\frac{\partial u}{\partial x}\right) + f(u),$$
$$(x,t) \in \Omega = \{(x,t): 0 < x < l,\ 0 < t \le T\}, \tag{1}$$

where $f(u)$ is the given function; $a(u) > 0$ is the piecewise continuous function defining the thermal conductivity coefficient, which is not given.

To identify the coefficient $a(u)$ under different initial and boundary conditions

$$u(x,0) = \varphi_0^i(x),\ x \in [0,l], \tag{2}$$

$$u(0,t) = \varphi_1^i(t),\ u(l,t) = \varphi_2^i(t),\ t \in [0,T], \tag{3}$$

$$i = 1,2,...,K,$$

observations of the dynamics of the object's state have been carried out:

$$u^i(x,t_j)=U^i_j(x),\ x\in[0,l],\ \ j=1,2,...,N^i, \quad (4)$$
$$i=1,2,...,K,$$

where

$$t_j\in[0,T],\ j=1,2,...,N^i$$

are given times of observation of the process state; N^i is the number of observations. It is possible to consider other forms of observations of the process state, for example, when there are observations of the process state not throughout the segment $[0,l]$, but at its separate points, i.e.:

$$u^i(x_m,t_j)=U^i_{mj},\ x_m\in[0,l],$$
$$m=1,2,...,M^i,\ j=1,2,...,N^i, \quad (5)$$
$$i=1,2,...,K.$$

We set the problem of determining the dependence of the thermal conductivity coefficient on the temperature, i.e. the function $a(u)$, on basis of observations of the process dynamics in the form (4) or (5). Such kinds of problems arise when studying qualitative properties of materials and environment which are impossible to measure directly, but via indirect measurements.

The quality of identification is estimated with the use of the least squares criterion. For each type of observation (4) or (5), the specific form of the criterion is different. To be definite, assume that there are observations of the form (4). Then the quality of identification is estimated as follows:

$$I(a)=\sum_{i=1}^{K}\sum_{j=1}^{N^i}\int_0^l\left[u^i(x,t_j;a)-U^i_j(x)\right]^2dx$$
$$+\varepsilon\|a(u)\|^2\to\min_{a(u)\in A}, \quad (6)$$

where A is given convex and closed domain of admissible values $a(u)$; $u^i(x,t;a)$ is the solution to the initial and boundary problem (1)–(3).

III. Problem Solution

To solve the problem (1)–(4), (6) and to obtain the function $a(u)$, we propose the following approach. Let, on the assumption of a priori information on the process, we know that

$$\underline{u}\le u(x,t)\le\overline{u},\ (x,t)\in\Omega, \quad (7)$$

where $\underline{u}$, $\overline{u}$ are known limit values of the process phase state. Let us quantize the set of phase state values by predetermined values u_l, $l=0,1,...,L$, where L is given:

$$\underline{u}=u_0<u_1<...<u_L=\overline{u}. \quad (8)$$

In particular, the quantization can be even:

$$u_{l+1}=u_l+(\overline{u}-\underline{u})/L,\ l=0,1,...,L-1,$$

and it can also depend on a priori information on the process. We shall assume that the function $a(u)$ is piecewise constant in the process's state space:

$$a(u)=a_{l+1},\ u_l\le u(x,t)<u_{l+1},$$
$$l=0,1,...,L-1 \quad (9)$$

or we can consider, for example, the case of piecewise linear functions; at that it is possible to approximate the function $a(u)$ as precisely as is required when $k\to\infty$. An important point here is the following matching condition: along the curve $\Gamma_{ks}(x,t)$ which consists of the points (x,t) at which the piecewise constant function $a(u)$ changes its value from a_k to a_s, we need to set additional "interfacing conditions":

$$u^+(x,t)\Big|_{\Gamma_{ks}(x,t)}=u^-(x,t)\Big|_{\Gamma_{ks}(x,t)}, \quad (10)$$

$$\left(a_k\frac{\partial u(x,t)}{\partial x}\right)^+\Bigg|_{\Gamma_{ks}(x,t)}=\left(a_s\frac{\partial u(x,t)}{\partial x}\right)^-\Bigg|_{\Gamma_{ks}(x,t)}. \quad (11)$$

Thus the problem of finding the function $a(u)$ is reduced to the problem of determining the finite-dimensional vector $a=(a_1,a_2,...,a_L)$ under conditions (1)-(4), (6)-(11).

To solve the formulated inverse problem, we propose to use first order finite-dimensional optimization methods [3]. With this end in view, we derive formulas for the gradient of the target functional in the space of optimizable parameters for both interval observations (4) and point observations (5).

Theorem

The components of the gradient of the functional of the problem (1)-(4), (6)-(11), in the space of identifiable values of the parameters $a\in R^L$, are determined by the following formulas:

$$\frac{\partial I(a)}{\partial a_l}=\sum_{i=1}^{K}\left[\iint_{(x,t)\in\Omega^i_l(a)}\psi^i(x,t)\cdot u^i_{xx}(x,t)dxdt\right]+2\varepsilon a_l,$$
$$l=1,2,...,L. \quad (12)$$

Here the set

$$\Omega^i_l(a)=\{(x,t)\in\Omega:\ u_{l-1}\le u^i(x,t;a)<u_l,$$
$$l=1,2,...,L\},i=1,2,...,K;$$

$\psi^i(x,t)$ is the solution to the following adjoint initial and boundary problem

$$\frac{\partial\psi}{\partial t} = -\frac{\partial}{\partial x}\left(a(u)\frac{\partial\psi}{\partial x}\right) - \\ -2\sum_{j=1}^{N^i}\delta(t-t_j)\int_0^l\left[u^i(x,t_j;a)-U_j^i(x)\right]dx, \quad (13)$$

$$\psi^i(x,T) = \psi^i(0,t) = \psi^i(l,t) = 0\,, \quad (14)$$

$$i = 1,2,...,K\,,$$

which satisfies the following additional matching conditions on the curves $\Gamma_{ks}(x,t)$:

$$\psi^+(x,t)\Big|_{\Gamma_{ks}(x,t)} = \psi^-(x,t)\Big|_{\Gamma_{ks}(x,t)}, \quad (15)$$

$$\left(a_k\frac{\partial\psi(x,t)}{\partial x}\right)^+\Bigg|_{\Gamma_{ks}(x,t)} = \left(a_s\frac{\partial\psi(x,t)}{\partial x}\right)^-\Bigg|_{\Gamma_{ks}(x,t)}. \quad (16)$$

We build numerical schemes of solution to the inverse problem considered on basis of the formulas (12)-(16). The values of the optimizable vector obtained are then used to build the function $a(u)$ from some class of functions. We decide on the form of this function on the basis of values $a(u) = a_l,\ u \in [u_{l-1}, u_l)$. For this purpose, we can make use of interpolation and approximation methods.

It is not hard to extend our approach to solution to coefficient inverse problems for quasilinear parabolic equations to other types of partial differential equations.

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