Stochastic Maximum Principle for Switching Systems

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Abstract— **This paper provides necessary conditions of optimality, in the form of a maximum principle, for optimal control problems of switching systems. Dynamics of the constituent processes take the form of stochastic differential equations with control terms in the drift and diffusion coefficients. The restrictions on the transitions or switches between operating modes, are described by collections of functional equality constraints.**

Keywords— Stochastic differential equation, Stochastic control system, Optimal control problem, Maximum principle, Switching system, Switching law.

I. INTRODUCTION

In general, a lot of real systems have abrupt changes in their dynamics that result from causes such as connections or disconnections of some components and success or failures in outcomes. These systems have stochastic behaviour and have been modeled by the class of stochastic differential equations [8,15,19].

Change of the structure of the system means that at some moment it may go over from one law of movement to another. After changing the structure, the characteristics of the initial condition of the system depends on its previous state. This situation joins them into a single system with variable structure [6].

A switching systems have the benefit of modeling dynamic phenomena with the continuous law of movement. Recently, optimization problems for switching systems have attracted a lot of theoretical and practical interest [6,9,16,18].

Stochastic control problems have a variety of practical applications in fields such as physics, biology, economics, management sciences, etc. [1,21]. The modern stochastic optimal control theory has been developed along the lines of Pontryagin's maximum principle and Bellman's dynamic programming [18]. The stochastic maximum principle has been first considered by Kushner [22]. Earliest results on the extension of Pontryagin's maximum principle to stochastic control problems are obtained in [7,10,12,20,22]. A general theory of stochastic maximum principle based on random convex analysis was given by Bismut [13]. Modern presentations of stochastic maximum principle with backward stochastic differential equations are considered in [14,23,24].

In this paper, backward stochastic differential equations have been used to establish a maximum principle for stochastic optimal control problems of switching systems. Such kind of problems have been considered by the authors in [2,3,5], where the optimal control problem of switching systems for stochastic systems with uncontrolled diffusion coefficients are studied. The problems with controlled diffusion coefficients without endpoint constraints are considered in [4].

In this paper, the optimal control problem of stochastic switching systems with control terms in the drift and diffusion coefficients and with endpoint constraints is considered. We obtain necessary condition of optimality in the form of a maximum principle for such systems, where the restrictions on transitions are described by equality constraints.

II. PRELIMINARIES AND STATEMENT OF PROBLEM.

Throughout this paper, we use the following notations. Let **N** be some positive constant, R^n denotes the n dimensional real vector space, $|.|$ denotes the Euclidean norm in R^n and E represents the mathematical expectation. Assume that $w_t^1, w_t^2, \dots, w_t^r$ are independent Wiener processes, which generate filtration $F_t^l = \bar{\sigma}(w_t^l, t_{l-1} \le t \le t_l),\ l = \overline{1,r}$, $0 = t_0 < t_1 < \dots < t_r = T$. Let $(\Omega, F, P), l = \overline{1,r}$ be a probability space with filtration $\{F_t, t \in [0,T]\}$, where $F_t = \bigcup_{l=1}^{r} F_t^l$. $L_F^2(a,b;R^n)$ stochastic denotes the space of all second order predictable processes $x_t(\omega)$.

$R^{m\times n}$ is the space of linear transformations from R^m to R^n . Let $O_l \subset R^{n_l}, Q_l \subset R^{m_l}, l = \overline{1,r}$, be open sets.

Consider the following stochastic control system:

$$dx_t^l = g^l\left(x_t^l, u_t^l, t\right)dt + f^l\left(x_t^l, u_t^l, t\right)dw_t \ \ t \in (t_{l-1}, t_l], \ l = \overline{1,r} \quad (1)$$

$$x_{t_{l-1}}^l = \Phi^{l-1}\left(x_{t_{l-1}}^{l-1} t_l\right) \quad l = \overline{2,r} \ ; x_{t_0}^1 = x_0, \quad (2)$$

$$u_t^l \in U_\partial^l \equiv \left\{u^l(t,\cdot) \in U^l \subset R^{m_l}, l = \overline{1,r} \quad a.c.\right\} \quad (3)$$

where $U^l, l = \overline{1,r}$ are non-empty bounded sets, and elements of $U_\partial^l, l = \overline{1,r}$ are called admissible controls. The problem is to find optimal inputs $\left(x^1, x^2, \ldots, x^r, u^1, u^2, \ldots, u^r\right)$ and switching sequence $t_1, t_2, \ldots, t_r$, such that the cost functional :

$$J(u) = \sum_{l=1}^{r} E\left[\varphi^l\left(x_{t_l}^l\right) + \int_{t_{l-1}}^{t_l} p^l\left(x_t^l, u_t^l, t\right)dt\right] \quad (4)$$

is minimized on the decisions of the system (1)-(3), which are generated by all admissible controls $U = U^1 \times U^2 \times \ldots \times U^r$ at conditions:

$$Eq^l\left(x_{t_r}^l\right) \in G, \quad l = \overline{1,r} \quad (5)$$

G is a closed convex set in R^k.

Assume that the following requirements are satisfied:

I. Functions $g^l, f^l, p^l, l = \overline{1,r}$ are twice continuously differentiable with respect to x.

II. Functions $g^l, f^l, p^l, \ l = \overline{1,r}$ and all their derivatives are continuous in (x,u). $g_x^l, g_{xx}^l, f_x^l, f_{xx}^l, p_{xx}^l$ are bounded

and hold the condition:

$$(1+|x|)^{-1}\left(\left|g^l(x,u,t)\right| + \left|g_x^l(x,u,t)\right| + \left|f^l(x,u,t)\right| + \left|f_x^l(x,u,t)\right| + \right.$$
$$\left. + \left|f_x^l(x,u,t)\right| + \left|p^l(x,u,t)\right| + \left|p_x^l(x,u,t)\right|\right) \le N.$$

III. Functions $\varphi^l(x): R^{n_l} \to R, l = \overline{1,r}$ are twice continuously differentiable and satisfy the condition:

$$\left|\varphi^l(x)\right| + \left|\varphi_x^l(x)\right| \le N(1+|x|), \ \left|\varphi_{xx}^l(x)\right| \le N.$$

IV Functions $\Phi^l(x,t): R^{n_l} \times T \to R^1, l = \overline{1,r-1}$ are continuously differentiable with respect to (x,t) and hold the condition:

$$\left|\Phi^l(x,t)\right| + \left|\Phi_x^l(x,t)\right| \le N(1+|x|).$$

V Functions $q^l(x): R^{n_l} \times R^1 \to R^1, l = \overline{1,r}$ are twice continuously differentiable and meet the condition:

$$\left|q^l(x)\right| + \left|q_x^l(x)\right| \le N(1+|x|).$$

Consider the sets: $A_i = \mathrm{T}^{i+1} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Lambda_j \times \prod_{j=1}^{i} Q_j, \ i = \overline{1,r},$ with the elements $\pi^i = (t_0, t_1, t_i, x_t^1, x_t^2, \ldots, x_t^i, u^1, u^2, \ldots, u^i)$.

III. MAXIMUM PRINCIPLE

To state the main result of this paper, we need to introduce the following theorem is proved in [4].

Theorem 1 Suppose that, conditions I-IV hold and $\pi^r = (t_0, t_1, t_r, x_t^1, x_t^2, \ldots, x_t^r, u^1, u^2, \ldots, u^r)$ is an optimal solution of problem (1)-(4). Then,

a) there exist random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ and $(\Psi_t^l, \mathrm{K}_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ which are the solutions of the following conjugate equations:

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t, \ \ t_{l-1} \le t < t_l, \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \psi_{t_r}^l = -\varphi_x^l(x_{t_r}^l), \ l = \overline{1,r}; \end{cases} \quad (6)$$

$$\begin{cases} d\Psi_t^l = -[g_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l + \Psi_t^l g_x^l(x_t^l, u_t^l, t) + \\ + f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)dt + f_x^{l*}(x_t^l, u_t^l, t)\mathrm{K}_t^l + \\ + \mathrm{K}_t^l f_x^l(x_t^l, u_t^l, t) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t)] \ dt + \mathrm{K}_t^l dw_t^l, \\ \Psi_{t_l}^l = -\varphi_{xx}^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_{xx}^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \Psi_{t_r}^l = -\varphi_{xx}^l(x_{t_r}^l) \end{cases} \quad (7)$$

b) $\forall \tilde{u}^l \in U^l, l = \overline{1,r}$, and a.e. $\theta \in [t_{l-1}, t_l]$ fulfills the maximum principle:

$$\begin{aligned} &H^l(\psi_\theta^l, x_\theta^l, u^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) + \\ &+ 0.5\Delta_{u^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Psi_\theta^l \Delta_{u^l} f^l(x_\theta^l, u_\theta^l, \theta) \le 0, \text{ a.c.} \end{aligned} \quad (8)$$

c) following transversality conditions hold.:

$$\psi_{t_l}^{l+1}\Phi_t^l\left(x_{t_l}^l, t_l\right) = 0, l = \overline{1, r-1}, a.c. \quad (9)$$

Here

$$H^l(\psi_t, x_t, u_t, t) = \psi_t g^l(x_t, u_t, t) + \beta_t f^l(x_t, u_t, t) - p^l(x_t, u_t, t),$$
$$t \in [t_{l-1}, t_l], \quad l = \overline{1,r}.$$

Then using the obtained result of the Theorem 1 and Ekeland's variational principle [17] the following theorem for stochastic optimal control problem of switching systems with constraints (5) is proved.

Theorem 2. Suppose that, conditions I-V hold and $\pi^r = (t_0, t_1, t_r, x_t^1, x_t^2, \ldots, x_t^r, u^1, u^2, \ldots, u^r)$ is an optimal solution of problem (1)-(5). Then,

a) there exist non-zero vectors $(\lambda_0, \lambda_1, \ldots, \lambda_r) \in R^{r+1}$ and random processes $(\psi_t^l, \beta_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ and $(\Psi_t^l, \mathrm{K}_t^l) \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l}) \times L_{F^l}^2(t_{l-1}, t_l; R^{n_l x n_l})$ which are the solutions of the conjugate equations:

$$\begin{cases} d\psi_t^l = -H_x^l(\psi_t^l, x_t^l, u_t^l, t)dt + \beta_t^l dw_t^l, \ \ t_{l-1} \le t < t_l, \\ \psi_{t_l}^l = -\varphi_x^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \psi_{t_r}^l = -\lambda_0 \varphi_x^l(x_{t_r}^l) - \sum_{l=1}^r \lambda_l q_x^l(x_{t_r}^l); \end{cases} \tag{10}$$

$$\begin{cases} d\Psi_t^l = -[g_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l + \Psi_t^l g_x^l(x_t^l, u_t^l, t) + \\ f_x^{l*}(x_t^l, u_t^l, t)\Psi_t^l f_x^l(x_t^l, u_t^l, t)dt + f_x^{l*}(x_t^l, u_t^l, t)\mathrm{K}_t^l + \\ \mathrm{K}_t^l f_x^l(x_t^l, u_t^l, t) + H_{xx}^l(\psi_t^l, x_t^l, u_t^l, t)] \ dt + \mathrm{K}_t^l dw_t^l, \\ \Psi_{t_l}^l = -\varphi_{xx}^l(x_{t_l}^l) + \psi_{t_{l+1}}^l \Phi_{xx}^l(x_{t_l}^l, t_l), \ l = \overline{1, r-1}, \\ \Psi_{t_r}^l = -\lambda_0 \varphi_{xx}^l(x_{t_r}^l) - \sum_{l=1}^r \lambda_l q_{xx}^l(x_{t_r}^l). \end{cases} \tag{11}$$

b) $\forall \tilde{u}^l \in U^l, l = \overline{1, r}$, a.e. $\theta \in [t_{l-1}, t_l]$ fulfills the maximum principle:

$$\begin{aligned} &H^l(\psi_\theta^l, x_\theta^l, \tilde{u}^l, \theta) - H^l(\psi_\theta^l, x_\theta^l, u_\theta^l, \theta) + \\ &+ 0.5\Delta_{\tilde{u}^l} f^{l*}(x_\theta^l, u_\theta^l, \theta)\Psi_\theta^l \Delta_{\tilde{u}^l} f^l(x_\theta^l, u_\theta^l, \theta) \le 0, \text{a.c.} \end{aligned} \tag{12}$$

c) following transversality conditions holds:

$$\psi_{t_l}^{l+1}\Phi_{t_l}^l\left(x_{t_l}^l, t_l\right) = 0, a.c., \ l = \overline{1, r-1} \quad (13).$$

Proof. Fist we discuss the existence of uniquely solutions of adjoint equations (10) and (11). In fact from [10,11,23,24], the first-order adjoint processes (ψ_t^l, β_t^l) and second order adjoint processes $(\Psi_t^l, \mathrm{K}_t^l)$ described in a unique way by (10) and (11) respectively. Finally, we obtain maximum principle in the case when and endpoint constraints are imposed.

For any natural j let's introduce the following approximating functional for each $l = \overline{1, r}$:

$$I_j^l\left(u^l\right) = S_j^l(E\varphi^l\left(x_{t_l}^l\right) + E\int_{t_{l-1}}^{t_l} p^l\left(x_t^l, u_t^l, t\right)dt, Eq^l(x_{t_r}^r)) =$$

$$\min_{c_j^l \in \varepsilon} \sqrt{\left|c_j^l - 1/j - E\left[\varphi^l(x_{t_l}^l) + \int_{t_{l-1}}^{t_l} p(x_t^l, u_t^l, t)dt\right]\right|^2 + \left|y - Eq^l(x_{t_r}^l)\right|^2}$$

,

where $\mathcal{E} = \{c : c \le J^0, y \in G\}$, J^0 minimal value of the functional in the problem (1)-(5).

It is easy to prove the following fact:

Lemma 1. Assume that conditions I-IV hold, $u_t^{l,n}, l = \overline{1, r}$ be the sequence of admissible controls from V^l, and $x_t^{l,n}$ be the sequence of corresponding trajectories of the system (1)-(3).

If the following condition is met: $d(u_t^{l,n}, u_t^l) \to 0$,

and then, $\lim_{n \to \infty}\left\{\sup_{t_{l-1} \le t \le t_l} E\left|x_t^{l,n} - x_t^l\right|^2\right\} = 0$,

where x_t^l is a trajectory corresponding to an admissible controls u_t^l , $l = \overline{1, r}$.

According to Ekeland's variational principle, there are controls such as; $u_t^{l,j} : d(u_t^{l,j}, u_t^l) \le \sqrt{\varepsilon_j^l}$ and for $\forall u_t^l \in V^l$ the following is achieved:

$$I_j^l(u^{l,j}) \le I_j^l(u^l) + \sqrt{\varepsilon_j^l}\, d(u^{l,j}, u^l), \ \varepsilon_j^l = \frac{1}{j}.$$

This inequality means that $(t_1, \ldots, t_r, x_t^{1,j}, \ldots, x_t^{r,j}, u_t^{1,j}, \ldots, u_t^{r,j})$ is a solution of the following problem:

$$\begin{cases} J_j(u \ ) = \sum_{l=1}^r \left(I_j^l(u^l) + \sqrt{\varepsilon_j^l}\, E\int_{t_{l-1}}^{t_l} \delta(u_t^l, u_t^{l,j})dt \right) \to \min \\ dx_t^l = g^l(x_t^l, u_t^l, t)dt + f^l(x_t^l, u_t^l t)dw_t, \quad t \in (t_{l-1}, t_l], \\ x_{t_l}^{l+1} = \Phi^l\left(x_{t_l}^l, t_l\right) \ \ l = \overline{1, r-1}\,; x_{t_0}^1 = x_0, \\ u_t^l \in U_\partial^l \end{cases} \tag{14}$$

Function $\delta(u, v)$ is determined in the following way:

$$\delta(u, v) = \begin{cases} 0, u = v \\ 1, u \ne v. \end{cases}$$

Then according to the Theorem 1, it is obtained as follows:

1) there exist the random processes $\psi_t^{l,j} \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l})$, $\beta_t^{l,j} \in L_{F^l}^2(t_{l-1}, t_l; R^{n_l \times n_l})$, which are solutions of the following system in $t \in [t_{l-1}, t_l)$:

$$\begin{cases} d\psi_t^{l,j} = -H_x^l\left(\psi_t^{l,j}, x_t^{l,j}, u_t^{l,j}, t\right)dt + \beta_t^{l,j} dw_t, \\ \psi_{t_l}^{l,j} = -\varphi_x^l\left(x_{t_l}^{l,j}\right) + \psi_{t_{l+1}}^l \Phi_x^l(x_{t_l}^{l,j}, t_l), \ l = \overline{1, r-1}; \\ \psi_{t_r}^l = -\lambda_0^j \varphi_x^l\left(x_{t_r}^{l,j}\right) - \sum_{l=1}^r \lambda_l^j q_x^l\left(x_{t_r}^{l,j}\right). \end{cases} \tag{15}$$

and the random processes $\Psi_t^{l,j} \in L^2_{F^l}(t_{l-1},t_l;R^{n_l})$, $K_t^{l,j} \in L^2_{F^l}(t_{l-1},t_l;R^{n_l\times n_l})$, which are solutions of the following system:

$$\begin{cases} d\Psi_t^{l,j} = -[g_x^{l*}(x_t^{l,j},u_t^{l,j},t)\Psi_t^{l,j} + \Psi_t^{l,j} g_x^l(x_t^{l,j},u_t^{l,j},t) + \\ + f_x^{l*}(x_t^{l,j},u_t^{l,j},t)\Psi_t^{l,j} f_x^l(x_t^{l,j},u_t^{l,j},t)dt + \\ + f_x^{l*}(x_t^{l,j},u_t^{l,j},t)K_t^{l,j} + K_t^{l,j} f_x^l(x_t^{l,j},u_t^{l,j},t) + \\ + H_{xx}^l(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t)]\ dt + K_t^{l,j}dw_t^l, \\ \Psi_{t_l}^{l,j} = -\varphi_{xx}^l(x_{t_l}^{l,j}) + \psi_{t_{l+1}}^{l,j}\Phi_{xx}^l(x_{t_l}^{l,j},t_l),\ l=\overline{1,r-1}, \\ \Psi_{t_r}^{l,j} = -\lambda_0^{l,j}\varphi_{xx}^l(x_{t_r}^{l,j}) - \sum_{l=1}^{r}\lambda_l^{l,j}q_{xx}^l(x_{t_r}^{l,j}). \end{cases} \qquad (16)$$

where non-zero $(\lambda_0^j,\lambda_1^j,\ldots,\lambda_r^j) \in R^{r+1}$ meet the following requirement:

$$\lambda = \left(\sum_{l=1}^{r}\left[-c_l + 1/j + E\varphi^l\left(x_{t_l}^{l,j}\right) + E\int_{t_{l-1}}^{t_l} p^l(x_t^{l,j},u_t^{l,j},t)dt\right],\right.$$

$$\left. Eq^1\left(x_{t_r}^{1,j}\right),\ldots,Eq^r\left(x_{t_r}^{r,j}\right)\right)/J_j^0\ \Bigg),\ \lambda = (\lambda_0^j,\lambda_1^j,\ldots,\lambda_r^j) \qquad (17)$$

2) almost certainly for $\forall \tilde{u}^l \in U^l$ $l=\overline{1,r}$, a.e. $t \in [t_{l-1},t_l]$ is satisfied:

$$H^l\left(\psi_t^{l,j},x_t^{l,j},\tilde{u}_t^l,t\right) - H^l\left(\psi_t^{l,j},x_t^{l,j},u_t^{l,j},t\right) + \\ 0.5\Delta_{\tilde{u}^l} f^{l*}(x_t^{l,j},u_t^{l,j},t)\Psi_t^{l,j}\Delta_{\tilde{u}^l} f^l(x_t^{l,j},u_t^{l,j},t) \le 0 \qquad (18)$$

3) the following transversality conditions hold:

$$\psi_{t_l}^{l+1,j}\Phi_{t_l}^l\left(x_{t_l}^{l,j}\right) = 0,\ l=\overline{1,r-1},\ a.c. \qquad (19).$$

Since the following has existed $\left\|(\lambda_0^j,\lambda_1^j,\ldots,\lambda_r^j)\right\| = 1$, then according to conditions I-IV it is implied that

$(\lambda_0^j,\lambda_1^j,\ldots,\lambda_r^j) \to (\lambda_0,\lambda_1,\ldots,\lambda_r)$ if $j\to\infty$.

Let us introduce the following results which will be needed in the future.

Lemma 2. Let $\psi_{t_l}^l$ be a solution of system (10), and $\psi_{t_l}^{l,j}$ be a solution of system (15). Then, for $j\to\infty$

$$E\int_{t_{l-1}}^{t_l}|\psi_t^{l,j} - \psi_t^l|^2\ dt + E\int_{t_{l-1}}^{t_l}|\beta_t^{l,j} - \beta_t^l|^2\ dt \to 0,\ l=\overline{1,r}\ ..$$

Lemma 3. Let $\Psi_{t_l}^{l,j}$ be a solution of system (11), and $\Psi_{t_l}^l$ be a solution of system (16). Then, for $j\to\infty$

$$E\int_{t_{l-1}}^{t_l}|\Psi_t^{l,j} - \Psi_t^l|^2\ dt + E\int_{t_{l-1}}^{t_l}|K_t^{l,j} - K_t^l|^2\ dt \to 0,\ l=\overline{1,r},$$

It follows from Lemma 2 and Lemma 3 that it can be proceeded to the limit in systems (15), (16) and the fulfilments of (10),(11) are obtained. Following the similar scheme by taking limit in (18) and (19) it is proved that (12), (13) are true. Theorem 2 is proved.

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