Asymptotic Analysis of RQ-systems M|M|1 on Heavy Load Condition

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***Abstract*— This paper reviews a RQ-system M|M|1. The research was carried out by the using asymptotic analysis method on heavy load condition. We have found analytical formula of asymptotic characteristic function for probability distribution of the number of requests at the source of repeated calls. In the paper comparison between numerical and asymptotic results are illustrated.**

Keywords — retrial queueing system; asymptotic analysis; heavy load

I. Introduction

The first subject of research in queuing theory are Erlang's telephone systems, characterized by a random stream of subscribers calls requiring random time for employment a telephone line [[1]].

In real information systems there are situations of repeated requirements to the service unit. One notable example is the lockout in terms of access to shared resources, in particular, conflicts for having access to files or blocks of memory, the lock of database objects in multiuser processing, etc. Mathematical model of the processes describing these systems is the RQ-system (Retrial Queueing System) [[2], [3]].

Many of the tasks for researching such systems have been solved numerically. In the paper an alternative way is used to solve them - a method of asymptotic analysis.

The method of asymptotic analysis in queuing theory is called the solution of equations systems determining the characteristics of the mathematical model, when some limiting conditions holds and these conditions has a specific form for each case.

II. Mathematical Model

Consider (Fig. 1) the single-line RQ-system with a source of repeated calls (orbit), which receives the input simple stream calls with parameter λ, and the service time of each application is exponentially distributed with parameter μ. If the application is received when the service device is free, it takes it for maintenance. If the device is busy, the request goes to orbit, where it performs a random delay, the duration of which has an exponential distribution with parameter σ. From the orbit after a random delay the application once again refers to the service unit to attempt to re-take it. If the device is free, the application is to serve by it, otherwise the application instantly returns to the source of repeated calls for the next delay implementation.

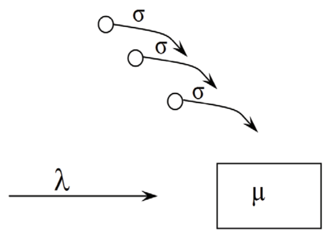

Figure 1. Single-line RQ-system

Let $i(t)$ is the number of requests in the orbit, and $k(t)$ is determines the status of the service device as follows:

$$k(t) = \begin{cases} 0, \text{ if it is free}, \\ 1, \text{ otherwise}. \end{cases}$$

We denote $P\{k(t)=k,\ i(t)=i\}=P(k,i,t)$ is the probability that at time t the device is in the state k and in the source of repeated calls there are i requests. The process $\{k(t),\ i(t)\}$ of system changing states over time is a Markov process.

For the probability distribution $P(k,i,t)$ of the states $\{k(t),\ i(t)\}$ for the RQ-system we make the direct system of Kolmogorov differential equations:

$$\begin{cases} \dfrac{\partial P(0,i,t)}{\partial t} = \mu P(1,i,t) - (\lambda + i\sigma)P(0,i,t) \\ \dfrac{\partial P(1,i,t)}{\partial t} = \lambda P(0,i,t) + (i+1)\sigma \cdot P(0,i+1,t) - (\lambda+\mu)P(1,i,t) + \\ \qquad\qquad + \lambda P(1,i-1,t). \end{cases} \tag{1}$$

In the stationary form the system (1) becomes:

$$\begin{cases} \mu P(1,i) - (\lambda + i\sigma)P(0,i) = 0, \\ \lambda P(0,i) + (i+1)\sigma \cdot P(0,i+1) - (\lambda+\mu)P(1,i) + \lambda P(1,i-1) = 0, \end{cases} \tag{2}$$

where $P(k,i,t) \equiv P(k,i)$.

In this system, turn to the characteristic functions:

$$H(k,u) = \sum_i e^{jui} P(k,i),$$

where $j=\sqrt{-1}$ is the imaginary unit.

Given the

$$\frac{\partial}{\partial u} H(k,u) = j\sum_i i e^{jui} P(k,i),$$

the system of equations (2) for the characteristic function can be rewritten as:

$$\begin{cases} \mu H(1,u)-\lambda H(0,u)=-\sigma j\dfrac{\partial H(0,u)}{\partial u}, \\ \lambda H(0,u)+\left(\lambda\left(e^{ju}-1\right)-\mu\right)H(1,u)=j\sigma e^{-ju}\dfrac{\partial H(0,u)}{\partial u}. \end{cases} \quad (3)$$

We introduce a parameter $\rho=\lambda/\mu$ characterizing the system load. The stationary regime in such a system exists for $\rho<1$.

Dividing by μ both sides of the equations (3) gives the following system:

$$\begin{cases} H(1,u)-\rho H(0,u)=-\dfrac{\sigma}{\mu}j\dfrac{\partial H(0,u)}{\partial u}, \\ \rho H(0,u)+\left(\rho\left(e^{ju}-1\right)-1\right)H(1,u)=j\dfrac{\sigma}{\mu}e^{-ju}\dfrac{\partial H(0,u)}{\partial u}. \end{cases} \quad (4)$$

In this paper we propose a method of asymptotic analysis for the solution of the system (4).

III. Investigation of the System

A. *Derivation of asymptotic equations*

The system (4) will be solved by the method of asymptotic analysis under heavy load, that is $\lambda/\mu=\rho\uparrow 1$, or when $\varepsilon\downarrow 0$, where $\varepsilon=1-\rho>0$ - an infinitely small quantity.

We introduce notations $u=\varepsilon w$, $H(0,u)=\varepsilon G(w,\varepsilon)$, $H(1,u)=F(w,\varepsilon)$. Then the system (4) can be rewritten as:

$$\begin{cases} F(w,\varepsilon)-(1-\varepsilon)\varepsilon G(w,\varepsilon)+j\dfrac{\sigma}{\mu}\dfrac{\partial G(w,\varepsilon)}{\partial w}=0, \\ (1-\varepsilon)\varepsilon G(w,\varepsilon)+\left((1-\varepsilon)\left(e^{jw\varepsilon}-1\right)-1\right)F(w,\varepsilon)-j\dfrac{\sigma}{\mu}e^{-jw\varepsilon}\dfrac{\partial G(w,\varepsilon)}{\partial w}=0. \end{cases} \quad (4)$$

Denote $F(w)=\lim\limits_{\varepsilon\to 0}F(w,\varepsilon)$ and $G(w)=\lim\limits_{\varepsilon\to 0}G(w,\varepsilon)$.

We write the following expansions of functions in Taylor series:

$$G(w,\varepsilon)=G(w)+\varepsilon g(w)+O(\varepsilon^2),$$

$$F(w,\varepsilon)=F(w)+\varepsilon f(w)+O(\varepsilon^2),$$

where $O(\varepsilon^2)$ - an infinitesimal quantity with order of ε^2.

By performing some action on the equations of the system (5) and taking the limit when $\varepsilon\downarrow 0$ the system can have next form:

$$\begin{cases} F(w)+j\dfrac{\sigma}{\mu}\dfrac{\partial G(w)}{\partial w}=0, \\ -F(w)+f(w)+\dfrac{\sigma}{\mu}w\dfrac{\partial G(w)}{\partial w}+j\dfrac{\sigma}{\mu}\dfrac{\partial g(w)}{\partial w}=0, \\ f(w)-G(w)+j\dfrac{\sigma}{\mu}\dfrac{\partial g(w)}{\partial w}=0. \end{cases} \quad (6)$$

The beforelimited characteristic function $H(u)=H(1,u)+H(0,u)$ under conditions of high load can be determined approximately by the equation: $H(u)\approx h(u)=F(w)$.

Thus, to solve the problem it is necessary to solve the system (6) for variable $F(w)$.

B. *Analysis of the asymptotic equations system*

Summarizing the 1st and 2nd equation and substituting in the third equation of (6) the differential equation of 1st order for $G(w)$ was obtained:

$$G(w)+\frac{\sigma}{\mu}(j+w)\frac{\partial G(w)}{\partial w}=0.$$

The general solution of this equation is:

$$G(w)=\left(\frac{c}{j+w}\right)^{\frac{\mu}{\sigma}},$$

where $c=const$.

Then

$$F(w)=-j\frac{\sigma}{\mu}\frac{\mu}{\sigma}\left(\frac{c}{j+w}\right)^{\frac{\mu}{\sigma}-1}\left(-\frac{c}{(j+w)^2}\right)=j\cdot C\cdot(j+w)^{-(\frac{\mu}{\sigma}+1)}, \quad (7)$$

where $C=const$.

As $h(0)\equiv 1$, where on heavy load $h(u)=F(w)$, therefore it can be assumed that $F(0)\equiv 1$.

The last identity can be used for determining the constant factor C in (7). So the following expression can be got:

$$F(w)=(1-jw)^{-\frac{\mu+\sigma}{\sigma}}.$$

Returning to the variable $u=\varepsilon w$ and parameter ρ. Then the characteristic function will be as follows:

$$h(u)=\left(1-j\frac{u}{1-\rho}\right)^{-\frac{\mu+\sigma}{\sigma}}.$$

The resulting expression is a characteristic function of γ-distribution $(1-\frac{ju}{\beta})^{-\alpha}$ with parameters $\alpha=\frac{\mu+\sigma}{\sigma}$ and $\beta=1-\rho$.

IV. Comparison of the Asymptotic and Numerical Distributions

We compare the resulting asymptotic distribution and the numerical solution of system (2).

The asymptotic distribution $P(i)$ which characteristic function is

$$h(u)=\left(1-j\frac{u}{1-\rho}\right)^{-\frac{\mu+\sigma}{\sigma}},$$

can be found by formula of Fourier transform

$$P(i)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-jui}\cdot h(u)du$$

or by using the properties of the γ-distribution with parameters $\alpha = \frac{\mu + \sigma}{\sigma}$ and β=1-ρ.

In the mathematical package MathCad the probability distribution of the number of requests in the orbit for various parameters λ, σ, ρ, μ was constructed. Also the difference between the numerical *R(i)* and asymptotic *P(i)* distributions was illustrated in the graphs. The Kolmogorov distance between the distributions was found:

$$\Delta = \max_{0 \le i < \infty} \left| \sum_{\nu=0}^{i} R(\nu) - \sum_{\nu=0}^{i} P(\nu) \right| .$$

Here are examples. Consider the case where ρ=0.9, μ=1, λ= ρ, σ=2. Fig. 2 shows the probability distributions of the number of requests in the orbit.

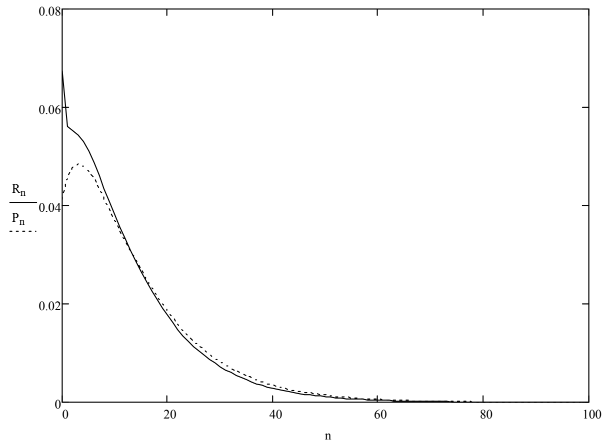

Figure 2. The difference between the numerical and asymptotic distributions where ρ=0.9, μ=1, λ= ρ, σ=2

On Fig. 3 the probability distributions of the number of requests in the orbit are presented for ρ=0.8, μ=1, λ= ρ, σ=1.

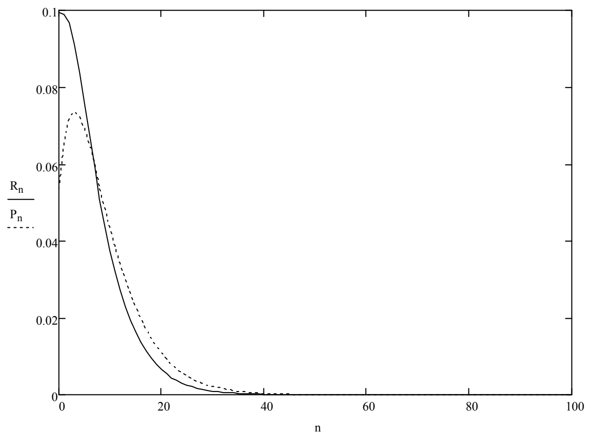

Figure 3. The difference between the numerical and asymptotic distributions where ρ=0.8, μ=1, λ= ρ, σ=1

In the last example we take ρ=0.95, μ=1, λ= ρ, σ=0.1. Fig. 4 shows the probability distributions of the number of requests in the orbit.

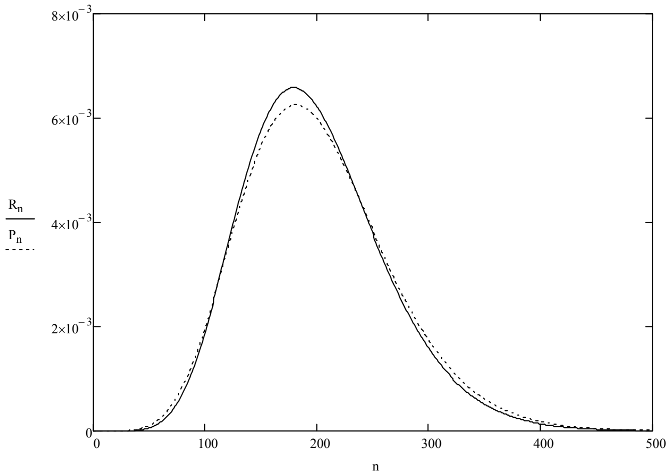

Figure 4. The difference between the numerical and asymptotic distributions where ρ=0.95, μ=1, λ= ρ, σ=0.1

In the following table the Kolmogorov distances between the numerical and asymptotic distributions for various parameters are given for different parameters values.

TABLE I. THE DIFFERENCE BETWEEN THE NUMERICAL AND ASYMPTOTIC DISTRIBUTIONS FOR VARIOUS PARAMETERS

Parameters	The Kolmogorov distance
ρ=0.9, μ=1, λ= ρ, σ=2	Δ=0.071
ρ=0.8, μ=1, λ= ρ, σ=1	Δ=0.141
ρ=0.8, μ=1, λ= ρ, σ=0.1	Δ=0.074
ρ=0.9, μ=1, λ= ρ, σ=1	Δ=0.055
ρ=0.9, μ=1, λ= ρ, σ=0.1	Δ=0.037
ρ=0.95, μ=1, λ= ρ, σ=1	Δ=0.046
ρ=0.95, μ=1, λ= ρ, σ=0.1	Δ=0.002

In this paper the mathematical model of RQ-system M|M|1 was be studied using the method of asymptotic analysis on the heavy load condition. The detailed derivation of asymptotic characteristic function for the resulting probability distributions of the number of requests at the source of repeated calls are presented in the article. The asymptotic distributions were compared with the numerical results, which showed a large area of the method application.

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