Investigation of High Intensive General Flow

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***Abstract*— The paper is devoted to investigation of general arrival process in condition of high intensity of events arrivals. It was proved that the process distribution can be approximated by normal distribution.**

***Keywords*— random process; arrival process; queueing theory**

Consider the general flow having independent and identically distributed inter-arrival period with length τ. We define distribution function for τ in the following way. Let $\tau = \xi / N$, where ξ is some nonnegative random variable with distribution function $A(z)$ and parameter N has meaning of great value (it's supposed that $N \to +\infty$). So

$$\mathrm{P}\{\tau\} = \mathrm{P}\{\xi / N < x\} = \mathrm{P}\{\xi < Nx\} = A(Nx).$$

Designate by $a = \mathrm{M}\{\tau\}$ – the average length of intervals between sequenced arrival events, by $b = 1 / a$ – the intensity of arrival events in the process and

$$b = 1 / \mathrm{M}\{\tau\} = 1 / \mathrm{M}\{\xi / N\} = N / \mathrm{M}\{\xi\} = N\lambda,$$

where

$$\lambda = \frac{1}{\mathrm{M}\{\xi\}} = \frac{1}{\int_0^{+\infty}(1 - A(z))\,dz}.$$

We have named this arrival process as *High Intensive General Independent* or *HIGI* arrival process because a great value N is present in its intensity.

Let's designate by $m(t)$ a number of arrival events in the process during the period of length t, and a length of period between a moment t and a moment when next event arrive by $z(t)$. Let's consider the two-dimension random process $\{m(t), z(t)\}$. Probability distribution of it values designate by

$$P(m, z, t) = \mathrm{P}\{m(t) = m, z(t) < z / N\}.$$

We will get the following expression for this distribution, if to use the full probability formula:

$$P(m, z, t + \Delta t) = P(m, z + N\Delta t, t) - P(m, N\Delta t, t) + P(m - 1, N\Delta t, t)\, A(z) + o(\Delta t)$$

from which we get Kolmogorov equation

$$\frac{1}{N}\frac{\partial P(m,z,t)}{\partial t} = \frac{\partial P(m,z,t)}{\partial z} - \frac{\partial P(m,0,t)}{\partial z} + A(z)\frac{\partial P(m-1,0,t)}{\partial z}, \tag{1}$$

here we use a designation:

$$\frac{\partial P(m,0,t)}{\partial z} = \left.\frac{\partial P(m,z,t)}{\partial z}\right|_{z=0}.$$

Summation of (1) over m from 0 to ∞ gives:

$$\frac{1}{N}\frac{\partial}{\partial t}\sum_{m=0}^{\infty}P(m,z,t) = \frac{\partial}{\partial z}\sum_{m=0}^{\infty}P(m,z,t) - \frac{\partial}{\partial z}\sum_{m=0}^{\infty}P(m,0,t) + A(z)\frac{\partial}{\partial z}\sum_{m=1}^{\infty}P(m-1,0,t). \tag{2}$$

Here

$$\sum_{m=0}^{\infty}P(m,z,t) = \mathrm{P}\left\{z(t) < \frac{z}{N}\right\}$$

is probability distribution of process $z(t)$. Let's designate its distribution in stationary mode by $R(z)$. And the following expression for $R(z)$ can be written:

$$0 = \frac{dR(z)}{dz} - \frac{dR(0)}{dz} + A(z)\frac{dR(0)}{dz}$$

or

$$\frac{dR(z)}{dz} = \frac{dR(0)}{dz}(1 - A(z)). \tag{3}$$

It has a solution

$$R(z) = \frac{dR(0)}{dz}\int_0^z (1 - A(x))dx\,.$$

It's obvious that $R(+\infty) = 1$, and we know that

$$\int_0^{+\infty}(1 - A(x))dx = \frac{1}{\lambda}.$$

As a result we get

$$\frac{dR(0)}{dz} = \lambda$$

and the solution to (3) rewrite as follow:

$$R(z) = \lambda\int_0^z (1 - A(x))dx. \tag{4}$$

Let's return to (1). We multiply it by $\exp\{jum\}$ where $j=\sqrt{-1}$ and u is some variable. Then sum it over m from 0 to ∞. If we designate

$$H(u,z,t)=\sum_{m=0}^{\infty} e^{jum}P(m,z,t),$$

then we get an equation for this function:

$$\frac{1}{N}\frac{\partial H(u,z,t)}{\partial t}=\frac{\partial H(u,z,t)}{\partial z}+\frac{\partial H(u,0,t)}{\partial z}\left(A(z)e^{ju}-1\right).$$

Making a substitution

$$H(u,z,t)=H_2(u,z,t)e^{juN\lambda t}$$

gives an equation for function $H_2(u, z, t)$:

$$\frac{1}{N}\frac{\partial H_2(u,z,t)}{\partial t}+ju\lambda H_2(u,z,t)= =\frac{\partial H_2(u,z,t)}{\partial z}+\frac{\partial H_2(u,0,t)}{\partial z}\left(A(z)e^{ju}-1\right) \tag{5}$$

This equation we shall solve by asymptotic analysis method [1]. After applying a designation $\varepsilon^2 = 1/N$ and substitutions $u=\varepsilon w$, $H_2(u, z, t) = F(u, z, t, \varepsilon)$ expression (5) gets a form

$$\varepsilon^2\frac{\partial F(w,z,t,\varepsilon)}{\partial t}+j\varepsilon w\lambda F(w,z,t,\varepsilon)= =\frac{\partial F(w,z,t,\varepsilon)}{\partial z}+\frac{\partial F(w,0,t,\varepsilon)}{\partial z}\left(A(z)e^{j\varepsilon w}-1\right) \tag{6}$$

Let's designate by

$$F(w,z,t)=\lim_{\varepsilon\to 0}F(w,z,t,\varepsilon).$$

Let's prove the following statement.

Theorem. The solution to (6) has the following form in condition of $\varepsilon\to 0$

$$F(w,z,t)=R(z)\exp\left\{\frac{(jw)^2}{2}(\lambda+\kappa)t\right\}, \tag{7}$$

where

$$\kappa = \lambda^3(\sigma^2 - a^2), \tag{8}$$

a and σ^2 is expected value and variance of random variable correspondingly, which distribution function is $A(x)$.

Prove the theorem at three stages.

Stage 1. Let $\varepsilon\to 0$ in (6), then we will get:

$$\frac{\partial F(w,z,t)}{\partial z}+\frac{\partial F(w,0,t)}{\partial z}\left(A(z)-1\right)=0.$$

This equation has a same form as (3), it follows that a function $F(u, z, t)$ can be written as

$$F(u, z, t) = R(z)\,\Phi(w, t) \tag{9}$$

where $\Phi(w, t)$ is some function.

Stage 2. We will search the solution to (6) as an expansion

$$F(u, z, t, \varepsilon) = \Phi(w, t)\,(R(z) + j\varepsilon w f(z)) + O(\varepsilon^2), \tag{10}$$

where $f(z)$ is some function. Let's substitute this in (6) and use expansion $\exp\{j\varepsilon w\} = 1 + j\varepsilon w + O(\varepsilon^2)$, we will get:

$$j\varepsilon w\lambda\Phi(w,t)R(z)=\Phi(w,t)\left[\frac{dR(z)}{dz}+j\varepsilon w\frac{df(z)}{dz}+\right. \left.+\left(\frac{dR(0)}{dz}+j\varepsilon w\frac{df(0)}{dz}\right)\left(A(z)-1+j\varepsilon wA(z)\right)\right]+O(\varepsilon^2).$$

Considering (3) and making simple transformations we get

$$\lambda R(z)=\frac{df(z)}{dz}+\frac{df(0)}{dz}\left(A(z)-1\right)+\lambda A(z)+O(\varepsilon).$$

Let $\varepsilon\to 0$, then we get a differential equation for unknown function $f(z)$:

$$\frac{df(z)}{dz}=\frac{df(0)}{dz}\left(1-A(z)\right)-\lambda\left(A(z)-R(z)\right).$$

Integration of the equation from 0 to +∞ gives

$$f(+\infty)-f(0)=\frac{df(0)}{dz}\cdot\frac{1}{\lambda}-\lambda\left(\int_0^{+\infty}[1-R(z)]dz-\int_0^{+\infty}[1-A(z)]dz\right). \tag{11}$$

Here $f(0) = 0$ because of (10) and simple probability condition

$$F(w,0,t,\varepsilon)=H(u,0,t)e^{-juN\lambda t}=e^{-juN\lambda t}\sum_{m=0}^{\infty}e^{jum}P(m,0,t)=0.$$

Let's calculate integral

$$\delta=\int_0^{+\infty}[1-R(z)]dz=\int_0^{+\infty}zdR(z).$$

Substitute (3) into this integral, we get

$$\delta = \lambda \int_0^{+\infty} z[1 - A(z)]dz = \frac{\lambda}{2} \int_0^{+\infty} z^2 dA(z)$$

and so (11) can be rewritten as

$$\frac{df(0)}{dz} - \lambda f(+\infty) = \frac{\lambda^3}{2} \int_0^{+\infty} z^2 dA(z) - \lambda = \frac{\kappa}{2}, \tag{12}$$

where κ is defined from (8).

Stage 3. Consider that

$$F(w,z,t,\varepsilon) = H(u,z,t)e^{-ju\lambda Nt} = e^{-ju\lambda Nt} \sum_{m=0}^{\infty} e^{jum} P(m,z,t),$$

where

$$\lim_{z\to+\infty} P(m,z,t) = P(m,t)$$

is a marginal distribution which doesn't depends on z and so

$$\lim_{z\to+\infty} \frac{\partial F(w,z,t,\varepsilon)}{dz} = 0.$$

Considering this and applying expansion

$$\exp\{j\varepsilon w\} = 1 + j\varepsilon w + (j\varepsilon w)^2/2 + O(\varepsilon^3)$$

in condition of $z \to +\infty$ we get from (6):

$$\varepsilon^2 \frac{\partial F(w,+\infty,t,\varepsilon)}{\partial t} + j\varepsilon w\lambda F(w,+\infty,t,\varepsilon) = \\ = \frac{\partial F(w,0,t,\varepsilon)}{\partial z}\left(j\varepsilon w + \frac{(j\varepsilon w)^2}{2} \right) + O(\varepsilon^3).$$

Substitution of (10) gives:

$$\varepsilon^2 \frac{\partial \Phi(w,t)}{\partial t} + j\varepsilon w\lambda \Phi(w,t) + (j\varepsilon w)^2 \lambda f(+\infty)\Phi(w,t) = \\ = \Phi(w,t)\left(j\varepsilon w\lambda + \frac{(j\varepsilon w)^2}{2}\lambda + (j\varepsilon w)^2 \frac{df(0)}{dz} \right) + O(\varepsilon^3).$$

After simple transforms we get:

$$\frac{\partial \Phi(w,t)}{\partial t} = (j\varepsilon w)^2 \Phi(w,t)\left(\frac{1}{2}\lambda + \frac{df(0)}{dz} - \lambda f(+\infty) \right) + O(\varepsilon).$$

Considering (12) and making $\varepsilon \to 0$ we get a differential equation for unknown function $\Phi(w, t)$:

$$\frac{\partial \Phi(w,t)}{\partial t} = \frac{(jw)^2}{2} \Phi(w,t)(\lambda + \kappa).$$

Boundary condition $\Phi(w, 0) = 1$ is followed from $P(m, z, 0) = R(z)$ if $m = 0$ and $P(m, z, 0) = 0$ otherwise.

So we get a solution

$$\Phi(w,t) = \exp\left\{ \frac{(jw)^2}{2}(\lambda + \kappa)t \right\}.$$

Considering (9) it follows that

$$F(w,z,t) = R(z)\exp\left\{ \frac{(jw)^2}{2}(\lambda + \kappa)t \right\}$$

and the proof.

When we return to function $H(u, z, t)$ and if value of N is great enough we have

$$H(u,z,t) \approx R(z)\exp\left\{ juN\lambda t + \frac{(ju)^2}{2} N(\lambda + \kappa)t \right\}.$$

Consider

$$h(u,t) = \lim_{z\to+\infty} H(u,z,t) -$$

characteristic function for a number of event arrivals $m(t)$ in HIGI arrival process. So, when value of N is great enough this function has a form of Gaussian process characteristic function:

$$h(u,t) \approx \exp\left\{ juN\lambda t + \frac{(ju)^2}{2} N(\lambda + \kappa)t \right\}$$

and it follows that distribution for $m(t)$ can be approximated by normal distribution with expected value equal to $N\lambda t$ and variance $N(\lambda + \kappa)t$.

Similar results were received for high intensive MAP flow also.

REFERENCES

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