Computational Aspects of the Method of Bilateral Narrowing of the Boundaries of the Optimum and the Variables in the Mixed-Integer Knapsack Problem

Kamal Mammadov
Ministry of Communications and Information Technologies, Baku, Azerbaijan
kamal_mamedov@yahoo.com, *ici-km@mincom.gov.az*

Abstract— **In this article experimental research was conducted in order to investigate the quality of the method for the solution of the problem of the mixed-integer programming with one restriction witch suggested by author. In these experiments it has been confirmed once more that the method suggested by author operates more rapidly than the known "branch and bound" method.**

Keywords*— *mixed-integer programming; integer-valued knapsack problem; interval narrowing; values of variables and functionals; equivalent problem

I. Introduction

A method of narrowing of the bilateral border, and functional variables in the following mixed-integer knapsack problem proposed in work [1]:

$$\sum_{j=1}^{N} c_j x_j \to \max \tag{1}$$

$$\sum_{j=1}^{N} a_j x_j \le b \tag{2}$$

$$0 \le x_j \le d_j ,\ (j = \overline{1, N}) \tag{3}$$

$$x_j,\ d_j \text{ are integers, } (j = \overline{1, n}),\ (n \le N) \tag{4}$$

Here, without losing generality, suppose that $c_j > 0$, $a_j > 0$, $d_j > 0$ ($j = \overline{1, N}$) and $b > 0$ are integers.

Without losing of generality, we suggest that in the problem (1) - (4), the unknowns are ordered as follows:

$$\frac{c_{j_{(1)}}}{a_{j_{(1)}}} \ge \frac{c_{j_{(2)}}}{a_{j_{(2)}}} \ge \ldots \ge \frac{c_{j_{(k)}}}{a_{j_{(k)}}} \ge \ldots \ge \frac{c_{j_{(N)}}}{a_{j_{(N)}}}$$

II. Problem Statement and Theoretical Ground of the Method

Let the optimal solution $\overline{X} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ of the continuous problem (1) – (3) without the condition of integrality of the variables, and an approximate solution to the integer part of the problem (1) - (4) found. Note that the solutions $\underline{X}$ and $\overline{X}$ are determined analytically, without any difficulties. Let's denote the K number of the unknowns, which takes on a fractional value in the optimal solution $\overline{X} = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_N})$.

Then, for the optimal value of f^* of the problem (1) – (4) we can define the upper ($\overline{f}$) and bottom ($\underline{f}$) boundaries as follows:

$$\overline{f} = \sum_{i=1}^{N} c_{j_{(i)}} \overline{x}_{j_{(i)}}, \quad \underline{f} = \sum_{i=1}^{N} c_{j_{(i)}} \underline{x}_{j_{(i)}} .$$

It is obvious that $\underline{f} \le f^* \le \overline{f}$.

Let $S_{j_{(i)}} = c_{j_{(i)}} - \dfrac{c_{j_{(k)}}}{a_{j_{(k)}}} a_{j_{(i)}}$, $i = \overline{1, N}$.

The following Theorem proved in work [1].

<u>Theorem 1.</u> Let $h_{j_{(i)}} = (\overline{f} - \underline{f}) / \left| S_{j_{(i)}} \right|$, $i = \overline{1, N}$. Then, the coordinates of the optimal solution of problem (1) – (4) change in the following interval:

a) if $i \in [1; k-1]$, then

$$x_{j_{(i)}} \in [\max\{0; d_{j_{(i)}} - [h_{j_{(i)}}]\}; d_{j_{(i)}}],$$

b) if $i \in [k+1; N]$, then

$$x_{j_{(i)}} \in [0; \min\{[h_{j_{(i)}}], d_{j_{(i)}}\}]$$

Here, $[z]$ represents the integer part of z.

Thus, as a result of the application of Theorem 1, the number of integer variables and the range of the feasible solutions containing the optimal solution decrease.

The following equivalent problem is obtained if we apply the Theorem 1 on problem (1) - (4):

$$\sum_{j=1}^{M} c_j y_j \to \max, \tag{5}$$

$$\sum_{j=1}^{M} a_j y_j \le b', \tag{6}$$

$$\alpha_j \le y_j \le d'_j, \quad j = \overline{1, M}, \tag{7}$$

$$y_j, d'_j \text{ are integers}, \ j = \overline{1, m}. \tag{8}$$

Here, $M \le N,\ m \le n,\ b' \le b$ and $y_j = x_j - \alpha_j$, $d'_j = \beta_j - \alpha_j,\ \ j = \overline{1, M}$.

The new values α_j and β_j $j = \overline{1, M}$ are determined after the application of Theorem 1.

As computing experiments show, the number of integer variables m in problem (5) - (8) is essentially less than n in original problem (1) - (4), i.e. $m << n$.

For finding the optimal solution of mixed-integer problem (5) - (8), in [1] we developed a new "branch and bound" type method in which, at each branching, the narrow intervals of values of the functional and the variable boundaries are essentially used.

It is necessary to note that, for every problem chosen from the list of unresolved problems, the region of feasible solutions is narrowed simultaneously, and the upper boundary of the functional decreases. It is clear that, due to such narrowing, the efficiency (i.e., the rate) of the method suggested in [1], will essentially exceed the usual "branch and bound" method. Numerous computing experiments prove this fact.

Note that the results received in [1], are the generalization of the results of works [2, 3] for a more general class of problems, namely, for the class of mixed-integer knapsack problems.

III. Results Of Numerical Experiments

To know about the efficiency of the suggested method in [1] in comparison with the known "branch and bound" method, a certain number of computing experiments have been executed. The factors of the solved problems were generated as pseudorandom numbers with uniform distributions from the interval $1 \le c_j \le 999$, $1 \le a_j \le 999$ and the integer, $j = \overline{1, N}$.

During the computing experiments, it was assumed that:

$$d_j = 20, \quad j = \overline{1, N}, \quad b = \left[0{,}3 \cdot \sum_{j=1}^{N} a_j \cdot d_j\right].$$

Let's note that, for each fixed dimension N and n, 8 problems were solved. The results are presented in the table 1 and 2, where:

N is the number of all the variables;

n is the number of the integer-valued variables;

Δ is the length of the variation of the intervals of the variables in original problem (1) - (3);

δ is the average length of the intervals for all the variables that remained after the application of Theorem 1. We shall note that, after the application of Theorem 1, the upper and the lower boundaries of many variables coincided. Naturally, the length of such intervals will be equal to zero. Therefore, among the mean values of δ, there are those smaller than unity.

P is the percent of the interval length shortening for the variables ($P = [(\Delta - \delta)/\Delta] \cdot 100$).

M and m are the mean value of the number of variable and integer-valued variables that remained after the application of Theorem (for 8 problems).

P_{avg} is the average percent of the decrease of the number of variables: $P_{avg} = \frac{N - M}{N} \cdot 100$;

P_{int} is the average percent of the decrease of the number of integer variables: $P_{\text{int}} = \frac{n - m}{n} \cdot 100$;

$t^b_{\min}$ and $t^b_{\max}$ are the minimal and maximal time of the solving of one problem taking into account 8 solved problems by the known "branch and bound" method, respectively;

t^b_{avg} is the average time of the solving of one problem taking into account 8 solved problems by the known "branch and bound" method;

$t_{\min}$ and $t_{\max}$ are the minimal and the maximal time of the solving of one problem taking into account 8 solved problems by the known "branch and bound" method suggested in [1].

t_{avg} is the mean time of the solving of one problem taking into account 8 solved problems by the [1].

TABLE I. RESULT OF 4 PROBLEMS THAT WERE SOLVED

N	200	200	500	500
n	100	150	250	350
Δ	20	20	20	20
δ	2,41	2,25	1,4	1,28
P	87,95	88,75	93	93,6
M	142	113	352	264
m	42	63	102	114
P_{int}	58%	58%	59,2%	67,4%
$t^b_{\min}$	8"	16"	2'34"	1'23"
$t^b_{\max}$	19'03"	24'15"	58'17"	59'03"
t^b_{avg}	4'27"	6'12"	18'41"	21'36"
$t_{\min}$	8"	15"	1'17"	1'23"
$t_{\max}$	3'35"	4'02"	10'23"	17'27"
t_{avg}	1'14"	1'46"	3'16"	4'16"

TABLE II. RESULT OF 4 PROBLEMS THAT WERE SOLVED

N	1000	1000	2000	2000
n	500	750	1000	1500
Δ	20	20	20	20
δ	0,83	0,78	0,64	0,62
P	95,85	96,1	96,8	96,9
M	654	460	1274	902
m	154	210	274	402
P_{int}	69,2%	72%	72,6%	73,2%
$t^b_{\min}$	1'36"	4'17"	27'36"	46'12"
$t^b_{\max}$	-	-	-	-
t^b_{avg}	-	-	-	-
$t_{\min}$	1'36"	2'07"	5'37"	14'04"
$t_{\max}$	24'18"	32'14"	46'08"	57'49"
t_{avg}	9'14"	12'18"	16'29"	23'07"

Let's note that, for the calculations of the solution of one problem of each dimension, given 1 hour. Therefore, the cell of the table where the time is not specified indicates that, for the problem of the corresponding dimension, there was required more than one hour.

IV. CONCLUSION

From the table, it is seen that, at the application of Theorem 2, the length of the initial intervals for the variables decreases on the average by 86 - 96% (this result corresponds to the results [1]). This means that, after the application of Theorem 1, it is possible to determine a considerably smaller part of the region of feasible solutions that contains the optimal solution.

From the table, it is also seen that the method developed in [1] runs much more quickly than the widely known "branch and bound" method. This is connected with two circumstances: 1 - is the number of variables and the region of feasible solutions are decreased; 2 - at each branching, the upper and the lower boundaries of the optimal functional value are narrowed simultaneously.

REFERENCES

[1] Mansimov K. B., Mamedov K. K. Mixed Integer Knapsack Problem Solving Method Using the Narrow Intervals for the Criterion Function and Variables. Automatic Control and Computer Sciences, 2010, Vol. 44, No. 4, pp. 216–226.

[2] Mamedov K. Sh. Definition of the narrow intervals for variables in the integer-valued Knapsack problem. J. "Applied and Computational Mathematics", 2003, v.2, № 2, pp. 156 - 160.

[3] Mamedov K. Sh., Mardanov S. S. The method of narrowing of the boundaries and the branch for the integer knapsack problem. J. Izv. NAS Azerbaijan, 2003, №2, pp. 82-87. (Russian)

[4] Mamedov K. Sh. A priory determination of the boundaries of the optimal value of variables in problems of integer linear programming. J. «Cybernetics and System Analysis», 2006, № 2, pp. 86-93. (Russian)