Numerical Method of Parametrical Identification for Nonlocal Parabolic Problems

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***Abstract*— The inverse problem for a parabolic equation with an unknown coefficient on the right-hand side is considered in this work. The dependence identifying coefficient only from variable of time and only from phase coordinates is investigated separately. The numerical method to solution of the problem by using the method of lines is suggested. The results of numerical experiments on test problems are given.**

***Keywords*— inverse problem; nonlocal conditions; method of lines; parabolic equation.**

I. INTRODUCTION

An attention to the boundary value problems with nonlocal conditions has increased in the last years. Nonlocal form of initial and boundary conditions is caused by practical impossibility to make measurements of the state of an object (or process) in its separate points or instantly in time. As a rule, the measured information reflects the state of a process in the neighbourhood of measurement point or during time interval of measurement, and sometimes it determines only mean value of the whole object and/or the mean value for the time interval of object's functioning [1,2].

In this work an approach to numerical solution is suggested both boundary value problems with nonlocal conditions, and inverse problems of parametrical identification for dynamic objects described by both a system of ordinary differential equations, and a parabolic equation.

II. PROBLEM FORMULATION

Let us consider the following parametrical identification problem, concerning the parabolic equation:

$$\frac{\partial v(x,t)}{\partial t} = a(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} + a_1(x,t)\frac{\partial v(x,t)}{\partial x} + a_2(x,t)v(x,t) + f(x,t) + B(t)C(x), \quad (1)$$

$$(x,t)\in\Omega = \{(x,t): 0<x<l,\ 0<t\le T\},$$

under initial, final and boundary conditions:

$$v(x,0) = \varphi_0(x), \quad x\in[0,l], \quad (2)$$

$$v(0,t) = \psi_0(t), \quad t\in[0,T], \quad (3)$$

$$v(l,t) = \psi_1(t), \quad t\in[0,T], \quad (4)$$

$$v(x,T) = \varphi_T(x), \quad x\in[0,l]. \quad (5)$$

Here functions $a = a(x,t) > 0, a_1(x,t), a_2(x,t), f(x,t), B(t)$, $\varphi_0(x), \varphi_T(x), \psi_0(t), \psi_1(t)$ are given and satisfy the following matching conditions:

$$\varphi_0(0) = \psi_0(0), \quad \varphi_0(l) = \psi_1(0), \quad \varphi_T(0) = \psi_0(T), \quad \varphi_T(l) = \psi_1(T),$$

as well as all the other conditions of existence and uniqueness of solution to the problem (1)-(5), which consists in determining the unknown function $C(x)$ and the corresponding solution to the boundary value problem $v(x,t)$ satisfying conditions (1)-(5).

For example, the following boundary value problem with nonlocal initial conditions is reduced to a particular case of the problem (1)-(5):

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial^2 u(x,t)}{\partial x^2} + \tilde{f}(x,t), \quad (x,t)\in\Omega, \quad (6)$$

$$k_1 u(x,0) + k_2\int_0^T e^{k\tau}u(x,\tau)d\tau = \varphi_0(x), \quad x\in[0,l], \quad (7)$$

$$u(0,t) = \tilde{\psi}_0(t), \quad u(l,t) = \tilde{\psi}_1(t), \quad t\in[0,T], \quad (8)$$

where k, k_1, k_2 are given constants, $\tilde{f}(x,t), \varphi_0(x)$, $\tilde{\psi}_0(t), \tilde{\psi}_1(t)$ are given functions.

To reduce the problem (6)-(8) to the problem (1)-(5) we introduce the function

$$v(x,t) = k_1 u(x,0) + k_2\int_0^t e^{k\tau}u(x,\tau)d\tau. \quad (9)$$

Differentiating (9) with respect to t, we obtain

$$\frac{\partial v(x,t)}{\partial t} = k_2 e^{kt}u(x,t), \quad (10)$$

whence it follows

$$u(x,t) = \frac{e^{-kt}}{k_2}\frac{\partial v(x,t)}{\partial t}. \quad (11)$$

From (7)-(9), (11), we obtain

$$v(x,0)=\frac{k_1}{k_2}\frac{\partial v(x,0)}{\partial t},\quad v(x,T)=\varphi_0(x),$$

$$v(0,t)=k_1\tilde{\psi}_0(0)+k_2\int_0^t e^{k\tau}\tilde{\psi}_0(\tau)d\tau=\psi_2(t),\qquad (12)$$

$$v(l,t)=k_1\tilde{\psi}_1(0)+k_2\int_0^t e^{k\tau}\tilde{\psi}_1(\tau)d\tau=\psi_1(t).$$

Differentiating (11) once with respect to t and twice with respect to x and taking it into account in the equation (6), after some transformations we obtain the following equation

$$\frac{\partial^2 v(x,t)}{\partial t^2}-a(x)\frac{\partial^3 v(x,t)}{\partial t\partial x^2}-k\frac{\partial v(x,t)}{\partial t}-k_2e^{kt}\tilde{f}(x,t)=0.\qquad (13)$$

Integrating both sides of (13) with respect to t, we obtain the following equality, accurate within the arbitrary function $C(x)$:

$$\frac{\partial v(x,t)}{\partial t}-a(x)\frac{\partial^2 v(x,t)}{\partial x^2}-kv(x,t)-f(x,t)=C(x),\qquad (14)$$

where $f(x,t)=k_2\int_0^t e^{k\tau}\tilde{f}(x,\tau)d\tau$. Unknown function $C(x)$ must be chosen so that conditions (12) are satisfied.

The problem (12), (14) coincides with the problem of parametrical identification (1)-(5) with the exception of the initial condition at $t=0$.

It is also possible to identify coefficient $B(t)$ instead of $C(x)$ in the problem (1)-(5). For example, the following problem with nonlocal boundary conditions is reduced to such a problem:

$$\frac{\partial u(x,t)}{\partial t}=a(t)\frac{\partial^2 u(x,t)}{\partial x^2}+\tilde{f}(x,t),\ (x,t)\in\Omega,\qquad (15)$$

$$\int_0^l e^{k\xi}u(\xi,t)d\xi=\psi(t),\quad t\in[0,T],\qquad (16)$$

$$u(0,t)=\psi_0(t),\quad t\in[0,T],\qquad (17)$$

$$u(x,0)=\tilde{\varphi}_0(x),\quad x\in[0,l].\qquad (18)$$

Suppose that the following matching condition is satisfied:

$$\int_0^l e^{k\xi}\varphi_0(x)d\xi=\psi(0).$$

Let us introduce the function

$$v(x,t)=\int_0^x e^{k\xi}u(\xi,t)d\xi.\qquad (19)$$

Differentiating (19):

$$\frac{\partial v(x,t)}{\partial x}=e^{kx}u(x,t),\qquad (20)$$

we obtain

$$u(x,t)=e^{-kx}\frac{\partial v(x,t)}{\partial x}.\qquad (21)$$

Differentiating (21) once with respect to t and twice with respect to x and taking it into account in (15), after simple transformations we obtain the following equation:

$$\frac{\partial^2 v(x,t)}{\partial t\partial x}=a(t)\frac{\partial^3 v(x,t)}{\partial x^3}-2a(t)k\frac{\partial^2 v(x,t)}{\partial x^2}+$$
$$+a(t)k^2\frac{\partial v(x,t)}{\partial x}+e^{kx}\tilde{f}(x,t).$$

Integrating both sides of the last equation with respect to x, we obtain the following equation, accurate within the arbitrary function $B(t)$:

$$\frac{\partial v(x,t)}{\partial t}=a(t)\frac{\partial^2 v(x,t)}{\partial x^2}-2a(t)k\frac{\partial v(x,t)}{\partial x}+$$
$$+a(t)k^2v(x,t)+f(x,t)+B(t),\qquad (22)$$

where $f(x,t)=\int_0^x e^{k\xi}\tilde{f}(\xi,t)d\xi$. From (16)-(21) we obtain the following initial and boundary conditions:

$$v(0,t)=0,\ \ t\in[0,T],\qquad (23)$$

$$v(l,t)=\psi(t),\ \ t\in[0,T],\qquad (24)$$

$$v(x,0)=\varphi_0(x),\ \ x\in[0,l],\qquad (25)$$

$$\frac{\partial v(0,t)}{\partial x}=\psi_0(t),\quad t\in[0,T],\qquad (26)$$

where $\varphi_0(x)=\int_0^x e^{k\xi}\tilde{\varphi}_0(\xi)d\xi$.

The obtained coefficient inverse problem (22)-(26) in a sense is a particular case of the problem (1)-(5). Note that the considered problem (1)-(5) itself is the subject of separate interest. Such problem may arise in the course of mathematical modeling and identifying parameters of some technological processes or dynamical behavior of objects with distributed parameters.

III. Numerical Solution to Problem of Determination of Function $C(x)$

For a numerical solution to the problem (1)-(5) it is possible to use various approaches. One of them consists in reducing it to an optimal control problem with the functional

$$J(C(x))=\int_0^l[v(x,T;C)-\varphi_T(x)]^2dx\to\min_{C(x)},$$

under conditions (1)-(4), which requires using iterative methods such as gradient type methods [3].

Another approach consists in constructing a fundamental solution to the problem (1)-(5) and reducing it to a problem with an integral equation. In the case when functions participating in the problem have a general form, using this approach faces basic difficulties.

A numerical method to solve the problem (1)-(5), based on using the method of lines [4-6] to reduce the problem to a system of ordinary differential equations with unknown parameters is suggested in this work. To determine unknown parameters we propose the analogue of a sweep method based on the results of the works [7-9].

Let us set up the lines $x_i = ih_x$, $i = 0,1,...,N$, $h_x = l/N$ in the domain Ω. On these lines we define the following functions:

$$v_i(t) = v(x_i, t),\quad t \in [0,T],\ i = 0,1,...,N,$$

which according to (2)-(5) entails:

$$v_i(0) = \varphi_0(x_i) = \varphi_{0i},\ i = 0,...,N, \tag{27}$$

$$v_0(t) = \psi_0(t),\quad t \in [0,T], \tag{28}$$

$$v_N(t) = \psi_1(t),\quad t \in [0,T], \tag{29}$$

$$v_i(T) = \varphi_T(x_i) = \varphi_{Ti},\ i = 0,...,N. \tag{30}$$

On the lines $x = x_i$ we approximate derivatives $\partial v/\partial x$, $\partial^2 v/\partial x^2$, by using central difference schemes:

$$\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=x_i} = \frac{v(x_i + h_x, t) - v(x_i - h_x, t)}{2h_x} + O(h_x^2) = \frac{v_{i+1}(t) - v_{i-1}(t)}{2h_x} + O(h_x^2),\ i = 1,...,N-1, \tag{31}$$

$$\left.\frac{\partial^2 v(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{v(x_i + h_x, t) - 2v(x_i, t) + v(x_i - h_x, t)}{h_x^2} + O(h_x^2) = \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_x^2} + O(h_x^2),\ i = 1,...,N-1. \tag{32}$$

Let us designate:

$$a_i(t) = a(x_i, t),\quad f_i(t) = f(x_i, t),\ a_{1i}(t) = a_1(x_i, t),$$
$$a_{2i}(t) = a_2(x_i, t),\quad C_i = C(x_i),\quad i = 1,...,N-1.$$

Taking into account (31), (32) in (1) we obtain system of $(N-1)$ th order ordinary differential equations involving unknown vector-parameters $C = (C_1,...,C_{N-1})^T$, which we write in a vector-matrix form:

$$\dot v(t) = A(t)v(t) + f(t) + B(t)C,\quad t \in (0,T], \tag{33}$$

$$v(0) = \varphi_0, \tag{34}$$

$$v(T) = \varphi_T, \tag{35}$$

where $v(t) = (v_1(t),...,v_{N-1}(t))^T$, $\varphi_0 = (\varphi_{01},...,\varphi_{0N-1})^T$, $\varphi_T = (\varphi_{T1},...,\varphi_{TN-1})^T$. The square tridiagonal $(N-1)\times(N-1)$ matrix A and the vector $f(t)$ are determined as follows:

$$A(t) = \frac{1}{h_x^2}\begin{pmatrix} -2a_1 + h_x^2 a_{21} & a_1 + \frac{h_x}{2}a_{11} & 0 & ... & 0 & 0 \\ a_2 - \frac{h_x}{2}a_{12} & -2a_2 + h_x^2 a_{22} & a_2 + \frac{h_x}{2}a_{12} & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_{N-1} - \frac{h_x}{2}a_{1N-1} & -2a_{N-1} + h_x^2 a_{2N-1} \end{pmatrix},$$

$$f(t) = \left(f_1 + \left(\frac{a_1}{h_x^2} - \frac{a_{11}}{2h_x}\right)\psi_0,\ f_2,...,f_{N-1} + \left(\frac{a_{N-1}}{h_x^2} - \frac{a_{1N-1}}{2h_x}\right)\psi_1\right)^T.$$

To determine vector C we use the analogue of a method for transferring conditions. Let a solution to the boundary value problem (33)-(35) be defined as follows:

$$v(t) = \alpha(t)C + \gamma(t), \tag{36}$$

where square $(N-1)\times(N-1)$ matrix function $\alpha(t)$ and $(N-1)$-dimensional vector function $\gamma(t)$ are arbitrary for a while.

Differentiating (36) and taking it into account in (33) we obtain:

$$\dot\alpha(t)C + \dot\gamma(t) = A(t)\alpha(t)C + A(t)\gamma(t) + f(t) + B(t)C.$$

The terms in the last expression can be grouped as follows:

$$[\dot\alpha(t) - A(t)\alpha(t) - B(t)E]C + [\dot\gamma(t) - A(t)\gamma(t) - f(t)] = 0,$$

where E is $(N-1)\times(N-1)$ identity matrix. Owing to the arbitrariness of functions $\alpha(t)$, $\gamma(t)$, we require from them the equality to zero of expressions in square brackets, from which we obtain the following systems of differential equations:

$$\dot\alpha(t) = A(t)\alpha(t) + B(t)E, \tag{37}$$

$$\dot\gamma(t) = A(t)\gamma(t) + f(t), \tag{38}$$

under initial conditions defined from (34), (36):

$$\alpha(0) = 0_{(N-1)\times(N-1)}, \tag{39}$$

$$\gamma(0) = \varphi_0, \tag{40}$$

where $0_{(N-1)\times(N-1)}$ is $(N-1)\times(N-1)$ zero matrix. Separately solving the matrix Cauchy problem (37), (39) and the Cauchy problem (38), (40) with respect to vector function $\gamma(t)$, by using condition (35) and representation (36), we obtain:

$$v(T) = \varphi_T = \alpha(T)C + \gamma(T). \tag{41}$$

(41) represents a system of $(N-1)$th order algebraic equations from which it is possible to determine identified vector C:

$$C = \alpha^{-1}(T)[\varphi_T - \gamma(T)].$$

Then, by applying any method of interpolation or approximation using the values of the components of vector $C = (C(x_1),...,C(x_{N-1}))$, it is possible to restore the unknown function $C(x)$ on the given class of functions.

IV. Numerical Solution to Problem of Determination of Function $B(t)$

Let us consider the problem of determination of function $B(t)$ from the differential equation (1) under conditions (23)-(26) which we write in a more general form:

$$v(x,0) = \varphi_0(x),\quad x \in [0,l], \tag{42}$$

$$v(0,t) = \psi_0(t),\quad t \in [0,T], \tag{43}$$

$$v(l,t) = \psi_1(t),\quad t \in [0,T], \tag{44}$$

$$\frac{\partial v(0,t)}{\partial x} = \psi_2(t),\quad t \in [0,T]. \tag{45}$$

Let us set up the lines in domain Ω:

$$t_j = jh_t,\ j = 0,1,...,N,\ h_t = T/N.$$

On these lines we define the following functions:

$$v_j(x) = v(x, t_j),\ \ x \in [0, l],\ j = 0,1,...,N,$$

which according to (42)-(45) entails:

$$v_0(x) = \varphi_0(x), \tag{46}$$

$$v_j(0) = \psi_0(t_j) = \psi_{0j},\quad j = 0,...,N, \tag{47}$$

$$v_j(l) = \psi_1(t_j) = \psi_{1j},\quad j = 0,...,N, \tag{48}$$

$$v'_j(0) = \psi_2(t_j) = \psi_{2j},\ j = 0,...,N. \tag{49}$$

On the lines $t = t_j$ we approximate derivative $\partial v(x,t)/\partial t$, by using difference scheme:

$$\left.\frac{\partial v(x,t)}{\partial t}\right|_{t=t_j} = \frac{v_j(x) - v_{j-1}(x)}{h_t} + O(h_t),\ j = 1,...,N. \tag{50}$$

Taking into account (50) in (1) we obtain the following system of N th order ordinary differential equations:

$$\begin{gathered} v''_j(x) + \tilde{a}_{1j}(x)v'_j(x) + \tilde{a}_{2j}(x)v_j(x) + \\ + \tilde{f}_j(x) + \tilde{C}(x)B_j = 0,\quad j = 1,...,N, \end{gathered} \tag{51}$$

where

$$B_j = B(t_j),\ \ \tilde{f}_j(x) = \frac{v_{j-1}(x) + h_t f(x,t_j)}{a(x,t_j)h_t},\ \ \tilde{C}(x) = \frac{C(x)}{a(x,t_j)},$$

$$\tilde{a}_{1j}(x) = \frac{a_1(x,t_j)}{a(x,t_j)},\quad \tilde{a}_{2j}(x) = \frac{a_2(x,t_j)h_t - 1}{a(x,t_j)h_t}.$$

Unlike the method of solving the problem of determination $C(x)$, explained in the previous section, the equations in (51) are sequentially solved from $j = 1$ to N and consequently vector components $B = (B_1,...,B_N)$ are also sequentially determined.

Suppose that on each line $t = t_j$ a solution to the problem is defined as follows:

$$v_j(x) = \alpha_j(x) + \beta_j(x)B_j, \tag{52}$$

where functions $\alpha_j(x)$, $\beta_j(x)$ are arbitrary for a while satisfying only condition (47), i.e.

$$\alpha_j(0) = \psi_{0j}, \tag{53}$$

$$\beta_j(0) = 0. \tag{54}$$

Differentiating (52) with respect to x, we determine $v'_j(x)$ and $v''_j(x)$:

$$v'_j(x) = \alpha'_j(x) + \beta'_j(x)B_j,\quad v''_j(x) = \alpha''_j(x) + \beta''_j(x)B_j. \tag{55}$$

From condition (49), taking into account (55) and the arbitrariness of functions $\alpha_j(x)$, $\beta_j(x)$, we require they satisfy the following conditions:

$$\alpha'_j(0) = \psi_{2j}, \tag{56}$$

$$\beta'_j(0) = 0. \tag{57}$$

Taking into account (55) in (51), after grouping we obtain:

$$\left[\alpha''_j(x) + \tilde{a}_{1j}(x)\alpha'_j(x) + \tilde{a}_{2j}(x)\alpha_j(x) + \tilde{f}_j(x)\right] +$$

$$+ \left[\beta''_j(x) + \tilde{a}_{1j}(x)\beta'_j(x) + \tilde{a}_{2j}(x)\beta_j(x) + \tilde{C}(x)\right]B_j = 0.$$

Owing to the arbitrariness of functions $\alpha_j(x)$, $\beta_j(x)$, we require from them the equality to zero of expressions in square brackets. We obtain the following differential equations:

$$\alpha''_j(x) + \tilde{a}_{1j}(x)\alpha'_j(x) + \tilde{a}_{2j}(x)\alpha_j(x) + \tilde{f}_j(x) = 0, \tag{58}$$

$$\beta''_j(x) + \tilde{a}_{1j}(x)\beta'_j(x) + \tilde{a}_{2j}(x)\beta_j(x) + \tilde{C}(x) = 0. \tag{59}$$

Solving the Cauchy problems (58), (56), (53) and (59), (57), (54), from (48), (52) we obtain

$$v_j(l) = \alpha_j(l) + \beta_j(l)B_j = \psi_{1j}.$$

From this, we obtain:

$$B_j = \frac{1}{\beta_j(l)}\left[\psi_{1j} - \alpha_j(l)\right]. \tag{60}$$

Thus to determine the components of vector $B = (B_1,...,B_N)$ it is necessary to solve N times the Cauchy problem concerning two independent second order differential equations. Then, the calculated vector $B = (B(t_1),...,B(t_N))$ can be used to obtain an analytical form of function $B(t)$ by applying methods of interpolation or approximation.

The carried-out numerous experiments on test problems have confirmed efficiency of the suggested approach. The results of various numerical experiments and their analysis are given in the report.

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