INVERSE TWO-PARAMETRIC PROBLEM WITH PERMUTATIONAL COMPACT SELF-ADJOINT OPERATORS UNDER RIGHT DEFINITENESS CONDITION

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Let it be known that each element of the given sequence $\{(\lambda_n, \mu_n)\}$ is an eigen value of some unknown spectral problem

$$\begin{cases} \lambda K_{i,1}\varphi_i + \mu K_{i,2}\varphi_i = \varphi_i, & \varphi_i \in H_i \\ i = 1;2 \end{cases}, \qquad (1)$$

where $K_{i1}, K_{i2}, i = 1;2$ are the desired compact self-adjoint permutational operators in Hilbert space H_i, $i = 1;2$ and let the given eigen element

$$\Phi_n = \varphi_{1n} \otimes \varphi_{2n} \in H = H_1 \otimes H_2 \text{ , } n = 1,2,..... . \qquad (2)$$

respond to the eigen value (λ_n, μ_n).

By $L(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in})$ denote a linear span of the set of the first n elements of the sequence $(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...)$, $i = 1;2$ i.e. of the set $\{\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}\} \subset H_i$, $i = 1;2$, and the closure of the linear subspace $L(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...)$ by $\overline{L}(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...)$, $i = 1;2$.

Introduce the denotation

$$\Delta_0 = K_{1,1} \otimes K_{2,2} - K_{1,2} \otimes K_{2,1},$$

$$\Delta_1 = I_1 \otimes K_{2,2} - K_{1,2} \otimes I_2 \text{ , } \Delta_2 = K_{1,1} \otimes I_2 - I_1 \otimes K_{2,1}.$$

$\Delta_0, \Delta_1, \Delta_2$ are linear operators determined on Hilbert space $H = H_1 \otimes H_2$ (a tensor product).

Lemma 1. Let the set (2) consist of all the eigen values of problem (1) and the right definiteness condition be fulfilled in problem (1) in the form

$$\Delta_0 = K_{1,1} \otimes K_{2,2} - K_{1,2} \otimes K_{2,1} > 0 \qquad (3)$$

Then, the closures of linear subspaces $L(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...)$, $i = 1;2$ coincide with spaces $H_i, i = 1;2$ respectively, i.e. $\overline{L}(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...) = H_i$, $i = 1;2$.

Proof. Let on the contrary, even if for one value of the index $i = 1;2$ the equality $\overline{L}(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...) = H_i$, $i = 1;2$ be not fulfilled. Then even if for one value of the index $i = 1;2$ there exists a subspace E_i that is invariant for a pair of operators $K_{i1}, K_{i2}, i = 1;2$ and it holds the equality $\overline{L}(\varphi_{i1}, \varphi_{i2}, ..., \varphi_{in}, ...) \oplus E_i = H_i$, $i = 1;2$. Therewith, even if for one value of the index $i = 1;2$ the equality $K_{i1}E_i = K_{i2}E_i = \{0\}$ should be fulfilled, i.e. there exists such $\varphi \in E_i$ that $K_{i1}\varphi = K_{i2}\varphi = 0$, consequently $(K_{i1}\varphi, \varphi) = (K_{i2}\varphi, \varphi) = 0$, This contradicts condition (3).

Corollary. If in problem (1) the right definiteness condition is fulfilled in the form (3), then the closure of the linear subspace $L(\Phi_1, \Phi_2, ..., \Phi_n, ...)$ coincides with the space $H = H_1 \otimes H_2$, i.e. $\overline{L}(\Phi_1, \Phi_2, ..., \Phi_n, ...) = H$.

Let $\varphi_i(\lambda, \mu) \in H_i, i = 1;2$ be a solution of the i-th equation of system (1) that analytically depends on the variables (λ, μ). The following theorem is known (see. [2]).

Theorem 1. In order all the eigen elements $\varphi_i(\lambda, \mu) \in H_i, i = 1;2$ that analytically depend on the parameters (λ, μ) be constants with respect to the variables (λ, μ), it is necessary and sufficient that the operators $K_{i,1}, K_{i,2}, i = 1;2$ in problem (1) be permutational. Therewith, the spectral lines will be straight lines.

Theorem 2. If K_{i1}, K_{i2} are the compact self-adjoint permutational operators in the Hilbert space $H_i, i = 1;2$ and the element $\varphi_i(\lambda, \mu) \in H_i, i = 1;2$ that analytically depends on the parameters (λ, μ), is an eigen element of the i-th equation of system (1), then $\varphi_i(\lambda, \mu) = \varphi_i \in H_i,\ i = 1;2$ (a constant with respect to (λ, μ)) is a joint eigen element of each of the operators K_{i1}, K_{i2} and the spectral line of the i-th equation of system (1) corresponding to the eigen element $\varphi_i \in H_i$, has the form $\lambda\, \alpha_{i1} + \mu\, \alpha_{i2} = 1, i = 1;2$, where $\alpha_{i1}, \alpha_{i2}\ i = 1;2$ are some eigen values of the operators $K_{i,1}, K_{i,2}, i = 1;2$ respectively.

Proof. Let $\varphi_i \in H_i, i = 1;2$ be an eigen element of the i-th equation of system (1) and by theorem 1 some spectral line in the form

$$\lambda\, c_{i1} + \mu\, c_{i2} = 1, i = 1;2 \tag{5}$$

corresponds to this eigen element.
Taking into account equality (5) in the system of equations (1), we get

$$\lambda\, (K_{i,1}\varphi_i - \frac{c_{i1}}{c_{i2}} K_{i,2}\varphi_i) = \varphi_i - \frac{1}{c_{i2}} K_{i,2}\varphi_i, \quad \varphi_i \in H_i \qquad i = 1;2 \tag{6}$$

Equality (6) is true for any value of the parameter λ. Here, the elements φ_i are independent of the parameter λ. Consequently, for any λ equality (6) is true iff the equalities

$$K_{i,1}\varphi_i - \frac{c_{i1}}{c_{i2}} K_{i,2}\varphi_i = 0, \quad \varphi_i - \frac{1}{c_{i2}} K_{i,2}\varphi_i = 0, \quad i = 1;2 ,$$

or $K_{i,1}\varphi_i = c_{i1}\varphi_i, \quad K_{i,2}\varphi_i = c_{i2}\varphi_i, \quad i = 1;2$ are fulfilled simultaneously. So, it is proved that the coefficients $c_{i1}, c_{i2},\ i = 1;2$ in equation (5) are the eigen values of the operators K_{i1}, K_{i2} that correspond to the joint eigen element $\varphi_i,\ i = 1;2$. The theorem is proved.

It is easily seen that if the set $\{e_{i1}, e_{i2}, ..., e_{in}, ...\} \subset H_i$ is an orthonormed basis of the space $H_i, i = 1;2$ consisting of all joint eigen elements of compact self-adjoint operators $K_{i,1}, K_{i,2}, i = 1;2$, then all possible decomposable tensors of the form $E_{n,m} = e_{1n} \otimes e_{2m} \in H = H_1 \otimes H_2$, $n, m = 1,2,....$ are the eigen elements of problem (1) that correspond to eigen values (λ_{nm}, μ_{nm}), where

$$\lambda_{nm} = \frac{\alpha_{22m} - \alpha_{12n}}{\alpha_{11n}\alpha_{22m} - \alpha_{12n}\alpha_{21m}}, \ \mu_{nm} = \frac{\alpha_{11n} - \alpha_{21m}}{\alpha_{11n}\alpha_{22m} - \alpha_{12n}\alpha_{21m}}, \tag{7}$$

$K_{i,j}e_{in} = \alpha_{ijn}e_{in}, \quad i, j = 1;2$ and vice verse, an arbitrary eigen element $\Phi \in H = H_1 \otimes H_2$ of problem (1) is a decomposable tensor $\Phi = \varphi_1 \otimes \varphi_2$, where $\varphi_i \in H_i$ is a joint eigen elemnt of the operators $K_{i,1}, K_{i,2}, i = 1;2$, i.e. if all the eigen values of permutational operators $K_{i,1}, K_{i,2}, i = 1;2$, are known, then the eigen values (λ_{nm}, μ_{nm}) of problem (1) are found by means of equalities (7).

Now, we can formulate a theorem that finds answer to the stated question on the inverse problem.

Theorem 3. Let the following three conditions be fulfilled:

1) $\{(\lambda_n, \mu_n)\}$ is a sequence of eigen elements of two-parametric problem (1)
2) $\{\Phi_n\} = \{\varphi_{1n} \otimes \varphi_{2n}\} \subset H = H_1 \otimes H_2$ is a sequence of appropriate eigen elements of problem (1)
3) it holds the equality $\overline{L}(\Phi_1, \Phi_2, ..., \Phi_n, ...) = H$,

Then there exists such a subsequence $\{n_k\} \subset N$ that $\{\varphi_{in_k}\} \subset H_i, \ i = 1;2$ is complete system of joint eigen elements of some permutational compact operators $K_{i,1}, K_{i,2}, i = 1;2$, where

$$K_{1,1} = \sum_{k=1}^{\infty} \frac{\mu_{n_{3k+1}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}} P_{1k}, \qquad K_{1,2} = \sum_{k=1}^{\infty} \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+1}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}} P_{1k},$$

$$K_{2,1} = \sum_{k=1}^{\infty} \frac{\mu_{n_{3k+2}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}} P_{2k}, \qquad K_{2,2} = \sum_{k=1}^{\infty} \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+2}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}} P_{2k},$$

and $(\lambda_{n_{3k}}, \mu_{n_{3k}}), (\lambda_{n_{3k+1}}, \mu_{n_{3k+1}}), (\lambda_{n_{3k+2}}, \mu_{n_{3k+2}})$ are the eigen values of problem (1) that correspond to three adjacent eigen elements $\Phi_{3k,3k} = \varphi_{1n_k} \otimes \varphi_{2n_k}$, $\Phi_{3k,3k+1} = \varphi_{1n_k} \otimes \varphi_{2n_{k+1}}$, $\Phi_{3k+1,3k} = \varphi_{1n_{k+1}} \otimes \varphi_{2n_k}$, respectively.

Here P_{ik} is a projection operator on to one-dimensional subspace $L\{\varphi_{in_k}\} \subset H_i$, $i = 1;2$

Proof. All possible tensor products in the form $\Phi_{k,r} = \varphi_{1n_k} \otimes \varphi_{2n_r}$ are eigen elements of problem (1). From the sequence $\Phi_{k,r} = \varphi_{1n_k} \otimes \varphi_{2n_r} \subset H = H_1 \otimes H_2$ we choose a subsequence $\{\Phi_{3k,3k}\} = \{\varphi_{1n_k} \otimes \varphi_{2n_k}\}_{k=1,2,....}$ in the following way:

a) $\Phi_{3,3} = \varphi_{11} \otimes \varphi_{21}$, i.e. $\varphi_{in_1} = \varphi_{i1}, i = 1;2$

в) $\Phi_{3(k+1),3(k+1)} = \varphi_{1n_{k+1}} \otimes \varphi_{2n_{k+1}}$, where $\varphi_{in_{k+1}} \notin L\{\varphi_{in_1}, \varphi_{in_2} ..., \varphi_{in_k}\} i = 1;2$.

Denote the eigen values that correspond to the eigen elements $\Phi_{3k,3k} = \varphi_{1n_k} \otimes \varphi_{2n_k}$ by $(\lambda_{n_{3k}}, \mu_{n_{3k}})$. And the eigen values that correspond to the eigen elements $\Phi_{3k,3k+1} = \varphi_{1n_k} \otimes \varphi_{2n_{k+1}}$ and $\Phi_{3k+1,3k} = \varphi_{1n_{k+1}} \otimes \varphi_{2n_k}$ denote by $(\lambda_{n_{3k+1}}, \mu_{n_{3k+1}})$ and

$\left(\lambda_{n_{3k+2}}, \mu_{n_{3k+2}}\right)$, respectively. To each eigen elements φ_{in_k}, of the i-th equation of system (1) there corresponds one spectral line $l_{i,k}$ in the form $\alpha_{in_k}\lambda + \beta_{in_k}\mu = 1$, $i = 1;2$. Consequently, to each eigen element $\Phi_{3k,3k} = \varphi_{1n_k} \otimes \varphi_{2n_k}$ of system (1) there corresponds a pair of spectral lines $l_{1,k}$ and $l_{2,k}$. The intersection of straight lines $l_{1,k}$ and $l_{2,k+1}$ will be $\left(\lambda_{n_{3k+1}}, \mu_{n_{3k+1}}\right)$, the intersection of straight lines $l_{1,k+1}$ and $l_{2,k}$ will be $\left(\lambda_{n_{3k+2}}, \mu_{n_{3k+2}}\right)$ (see fig. 1).

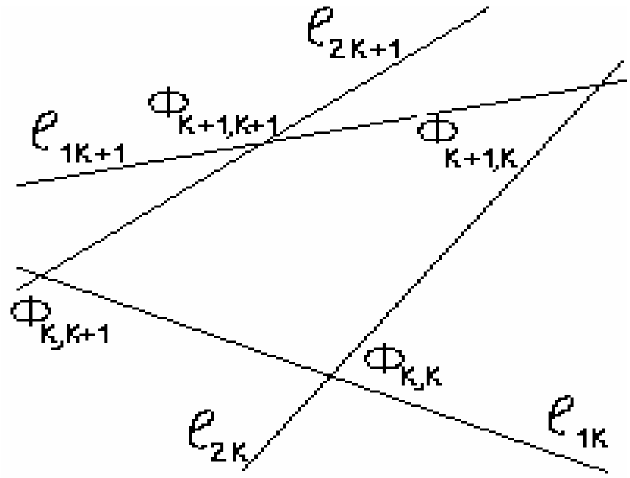

Fig. 1

Then the following equalities hold

$$\begin{cases}\alpha_{1n_k}\lambda_{n_{3k}} + \beta_{1n_k}\mu_{n_{3k}} = 1\\ \alpha_{1n_k}\lambda_{n_{3k+1}} + \beta_{1n_k}\mu_{n_{3k+1}} = 1\end{cases} \qquad \begin{cases}\alpha_{2n_k}\lambda_{n_{3k}} + \beta_{2n_k}\mu_{n_{3k}} = 1\\ \alpha_{2n_k}\lambda_{n_{3k+2}} + \beta_{2n_k}\mu_{n_{3k+2}} = 1\end{cases}$$

From the last system of equations we get

$$\alpha_{1k} = \frac{\mu_{n_{3k+1}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}}, \qquad \beta_{1n_k} = \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+1}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}},$$

$$\alpha_{2k} = \frac{\mu_{n_{3k+2}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}}, \qquad \beta_{2n_k} = \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+2}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}},$$

Consequently the following expansions hold for the compact self-adjoint operators $K_{i,1}, K_{i,2}, i = 1;2$:

$$K_{1,1} = \sum_{k=1}^{\infty} \frac{\mu_{n_{3k+1}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}} P_{1k}, \qquad K_{1,2} = \sum_{k=1}^{\infty} \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+1}}}{\lambda_{n_{3k}}\mu_{n_{3k+1}} - \lambda_{n_{3k+1}}\mu_{n_{3k}}} P_{1k},$$

$$K_{2,1} = \sum_{k=1}^{\infty} \frac{\mu_{n_{3k+2}} - \mu_{n_{3k}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}} P_{2k}, \qquad K_{2,2} = \sum_{k=1}^{\infty} \frac{\lambda_{n_{3k}} - \lambda_{n_{3k+2}}}{\lambda_{n_{3k}}\mu_{n_{3k+2}} - \lambda_{n_{3k+2}}\mu_{n_{3k}}} P_{2k},$$

where P_{ik} is a projectional operators on to one-dimensional subspace $L\{\varphi_{in_k}\} \subset H_i$, $i = 1;2$. The completeness of the system of eigen elemnts $\{\varphi_{in_k}\} \subset H_i$, $i = 1;2$ is easily obtained from the condition $\overline{L}(\Phi_1, \Phi_2, ..., \Phi_n, ...) = H$. The theorem is proved.

References

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