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SOLUTION OF SOME INVERSE PROBLEM OF ATMOSPHERIC OPTICS

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Let's consider the following coefficient inverse problem in general setting:

$$\sum_{i=1}^{n} a_{ik}\delta\alpha_i = -b_k \;, \tag{1}$$

where $a_{ik} = \left(\varphi^*_{p_k},\, A_i f' - \xi_i\right)$, $b_k = \delta I_{p_k}$, $k = 1,2,\ldots,m.$ (2)

$\varphi^*_{p_k}$ is a solution of adjoint integral transport equation in spherical atmosphere, A_i are known operators, ξ_i are known functions, f' is known solution of perturbed integral transport equation in spherical atmosphere, α_i, $i = 1,2,\ldots,n$ is a set of unknown parameters (coefficients), $\delta\alpha_i$ is a perturbation of operator α_i, $\alpha'_i = \alpha_i + \delta\alpha_i$, I_{p_k} is a functional (integral) which depends on radiation intensity, δI_{p_k} is a perturbation of I_{p_k}, $I'_p = I_p + \delta I_p$. [1], [2].

Let $f = Kf + \psi$ is an integral transport equation in spherical atmosphere. As functionals of the problem we consider readouts of devices, that measures a radiation intensity

$$I_p(f) = \int_D \int_{\Delta\Omega} f \cdot \xi \cdot \delta(\mathbf{r} - \mathbf{r}_0) d\mathbf{r} d\mathbf{\Omega} \,,$$

here ξ is known instrument function.

Let consider optical thicknesses $\tau_i(\mathbf{r}_n,\, \mathbf{r}_k,\, \lambda) = \int_0^{l_k} \sigma(\mathbf{r}_n + \boldsymbol{\omega}_k l,\, \lambda) dl$ as unknown parameters in different atmospheric layers and assume, that they are constants in indicated layers, where $\boldsymbol{\omega}_k = (\mathbf{r}_k - \mathbf{r}_n)/|\mathbf{r}_k - \mathbf{r}_n|$ is an optical path length from $\mathbf{r}_n$ to $\mathbf{r}_k$.

So, it's required to find $\boldsymbol{\sigma} \equiv \boldsymbol{\tau}$, $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_n)$. Suppose that a radiation is monochromatic. Expanding $\delta(f - Kf)$ as a Taylor's series on $\boldsymbol{\sigma}$ near the $\boldsymbol{\sigma}_0$ point, we have $\delta(f - Kf) \approx \sum_{i=1}^{n} \frac{\partial}{\partial\sigma_i}(f - K'f)\big|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}_0} \delta\sigma_i$ and hence

$$\left(f^*_{p_k},\, \sum_{i=1}^{n} \frac{\partial}{\partial\sigma_i}(f - K'f)\big|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}_0} \delta\sigma_i \right) = \sum_{i=1}^{n} \frac{\partial I_k}{\partial\sigma_i}\, \delta\sigma_i \,. \tag{3}$$

In this case system (1) with coefficients (2) has a form:

$$\sum_{i=1}^{n} \frac{\partial I_k}{\partial\sigma_i}\delta\sigma_i = -\delta I_{p_k} \quad \text{or}$$

$$\mathbf{A}\delta\boldsymbol{\sigma} = \mathbf{b} \,. \tag{4}$$

Obtained system (4) is inconsistent $(m > n)$, and we'll look for the most appropriate values for unknown quantities $\delta\sigma_i$. From the theory of the least squares, if we minimize $\|\mathbf{A}\delta\boldsymbol{\sigma} - \mathbf{b}\|^2 = \varepsilon^2$, then we'll have a system of normal equations

$$\mathbf{A}^*\mathbf{A}\delta\boldsymbol{\sigma} = \mathbf{A}^*\mathbf{b} \,. \tag{5}$$

Let's demand normal distribution for errors in measurement of I_p with a small difference between each other. Otherwise we can bring in statistical weights. Thereto, equations can be multiplied by quantities, that are inversely proportional to mean-square deviations of measured quantities. As far as relative mean-square deviations of functional estimates usually don't change strongly, then we can use also values of functionals as statistical weights;

$$\mathbf{A}^{*}\mathbf{W}\mathbf{A}\delta\boldsymbol{\sigma}=\mathbf{A}^{*}\mathbf{W}\mathbf{b}. \tag{6}$$

System of linear equations (6) can be solved with any of known methods.

In many cases system (6) turns out ill-conditioned, therefore in order to solve it, regularization methods by A.N. Tihonov should be applied [3]. The main idea of this method consists in minimization of the following expression:

$$\left\|\mathbf{A}^{*}\mathbf{W}\mathbf{A}\delta\boldsymbol{\sigma}-\mathbf{A}^{*}\mathbf{W}\mathbf{b}\right\|+\gamma\cdot\left(\delta\boldsymbol{\sigma},\ \Omega\delta\boldsymbol{\sigma}\right)=\varepsilon,$$

where Ω is an matrix approximation for $\int_0^H\left|\sum_{k=1}^{m}q_k(x)\frac{d^k\sigma}{dx^k}\right|dx$.

Here H is a height of atmosphere, $q_k(x)>0$. The last expression means that additional constraints imposed to the class of solutions. Usually $k=1$, $q_k(x)\equiv 1$. It means that limited derivative of the solution, $\delta\boldsymbol{\sigma}(x)$, is required. Regularization coefficient γ can be found approximately. In [4] the following value is presented:

$$\gamma=\frac{n}{\left(\delta\boldsymbol{\sigma},\ \Omega\delta\boldsymbol{\sigma}\right)}. \tag{7}$$

Where n is a number of dimensions. But as γ depends on unknown solution, it's offered to take the right-hand sides of (6) instead of $\delta\boldsymbol{\sigma}$ in (7). Then, we have $\gamma=\frac{n}{\left(\mathbf{A}^{*}\mathbf{W}\mathbf{b},\ \Omega\mathbf{A}^{*}\mathbf{W}\mathbf{b}\right)}$.

For estimation of functional $I_\varphi=\left(f,\varphi\right)=\int_X f\left(\mathbf{x}\right)\varphi\left(\mathbf{x}\right)d\mathbf{x}=\sum_{i=0}^{\infty}\left(K^i\psi,\varphi\right)$ and its derivative the algorithm of dependent tests of Monte Carlo methods is usually used. It follows that for estimation of I_φ by Monte Carlo methods it is required to average the sums of $\varphi\left(\mathbf{x}\right)$ evaluated from collisions with different order. The algorithm of dependent tests for transport theory problems consists in modeling of particles' trajectories in different systems by the same trajectories. Arising displacements are eliminated with special weight coefficients. Let λ is a wavelength and parameter of the system, that is $k\left(\mathbf{x},\mathbf{x}'\right)=k\left(\mathbf{x},\mathbf{x}',\lambda\right)$, $\varphi\left(\mathbf{x}\right)=\varphi\left(\mathbf{x},\lambda\right)=\varphi_\lambda$. Then trajectories constructed for $\lambda=\lambda_0$ can be applied for estimates of I_φ if after each passage $\mathbf{x}\to\mathbf{x}'$ auxiliary weight of particle is multiplied by $\frac{k\left(\mathbf{x},\mathbf{x}',\lambda\right)}{k\left(\mathbf{x},\mathbf{x}',\lambda_0\right)}$.

Now we'll show how to calculate $\frac{\partial I_k}{\partial\alpha_i}$ in (4). Let $I_k=I_k(t)$ depends on some parameter t. Then

$$I_k(t)=\sum_{n=0}^{\infty}\int\ldots\int\psi\left(\mathbf{x}_0\right)\prod_{i=0}^{n-1}k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)\prod_{i=0}^{n-1}\frac{k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t,\lambda\right)}{k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)}\varphi_k\left(\mathbf{x}_n,t,\lambda\right)d\mathbf{x}_0\ldots d\mathbf{x}_n.$$

Suppose that the last series can be termwise differentiated and differentiation can be done under the integral signs with corresponding dimensions. Then formally:

$$\frac{\partial}{\partial t}\left\{\psi\left(\mathbf{x}_0\right)\prod_{i=0}^{n-1}k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)\prod_{i=0}^{n-1}\frac{k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t,\lambda\right)}{k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)}\cdot\varphi\left(\mathbf{x}_n,t,\lambda\right)\right\}\Bigg|_{t=t_0}=$$

$$\psi\left(\mathbf{x}_0\right)\left(\prod_{i=0}^{n-1}k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)\right)Q_n\left(\lambda,t_0\right)\varphi_k\left(\mathbf{x}_n,t_0,\lambda\right)\left\{\frac{\partial\ln\varphi_k\left(\mathbf{x}_n,t,\lambda\right)}{\partial t}+\frac{\partial\ln Q_n\left(\lambda,t\right)}{\partial t}\right\}\Bigg|_{t=t_0}=$$

$$\psi\left(\mathbf{x}_0\right)\left(\prod_{i=0}^{n-1}k\left(\mathbf{x}_i,\mathbf{x}_{i+1},t_0,\lambda_0\right)\right)Q_n\left(\lambda,t_0\right)\varphi_k\left(\mathbf{x}_n,t_0,\lambda\right)\Psi_n\left(\lambda,t_0\right), \tag{8}$$

where $\Psi_n(\lambda,t)=\left\{\frac{\partial \ln \varphi_k(\mathbf{x}_n,t,\lambda)}{\partial t}+\sum_{i=0}^{n-1}\frac{\partial \ln k(\mathbf{x}_i,\mathbf{x}_{i+1},t,\lambda)}{\partial t}\right\}$.

For $t=\tau_i(\mathbf{r}_n,\ \mathbf{r}_k,\ \lambda)$ it can be shown that obtained series after differentiation under the integral sign has a absolutely convergent majorant , independent on $\tau_i(\mathbf{r}_n,\ \mathbf{r}_k,\ \lambda)$.

Derivatives are evaluated from local estimate with the following physics characteristics: $\sigma_a(\mathbf{r},\lambda)$ is a scattering aerosol cross-section with indicatrix $g_a(\mu,\mathbf{r},\lambda)$, $\sigma_m(\mathbf{r},\lambda)$ is a scattering aerosol cross-section with indicatrix $g_m(\mu,\mathbf{r},\lambda)$, $\sigma_c(\mathbf{r},\lambda)$ is an absorption cross-section, $\mu(\boldsymbol{\omega},\boldsymbol{\omega}')$ is a cosine of angle between the previous and next particle's directions in collision point, $\sigma(\mathbf{r},\lambda)=\sigma_a(\mathbf{r},\lambda)+\sigma_m(\mathbf{r},\lambda)$ is a total cross-section, $g(\mu,\mathbf{r},\lambda)=\frac{g_a(\mu,\mathbf{r},\lambda)\sigma_a(\mathbf{r},\lambda)+g_m(\mu,\mathbf{r},\lambda)\sigma_m(\mathbf{r},\lambda)}{\sigma(\mathbf{r},\lambda)|\mathbf{r}_n-\mathbf{r}_k|}$ is a full indicatrix, $\tau(\mathbf{r}_n,\mathbf{r}_k,\lambda)=\int_0^{l_k}\sigma(\mathbf{r}_n+\boldsymbol{\omega}_k l,\lambda)dl,\ \boldsymbol{\omega}_k=(\mathbf{r}_k-\mathbf{r}_n)/|\mathbf{r}_k-\mathbf{r}_n|$ is an optical path length from $\mathbf{r}_n$ to $\mathbf{r}_k$.

In this case, in order to estimate the radiation intensity integral on the directions in the given point $\mathbf{r}^*$ we use

$$\varphi_k^*=\varphi_k^*(\mathbf{r}_n,\boldsymbol{\omega}_n^*,\lambda)=c\cdot\frac{\exp(-\tau(\mathbf{r}_n,\mathbf{r}^*,\lambda))(g_a(\mu^*,\mathbf{r},\lambda)\sigma_a(\mathbf{r},\lambda)+g_m(\mu^*,\mathbf{r},\lambda)\sigma_m(\mathbf{r},\lambda))}{\sigma(\mathbf{r}_n,\lambda)}$$

and

$$k(\mathbf{x}_i,\mathbf{x}_{i+1},\lambda)=c_2\cdot\frac{\sigma(\mathbf{r}_{i+1},\lambda)}{\sigma(\mathbf{r}_i,\lambda)}\cdot\exp((-\tau(\mathbf{r}_i,\mathbf{r}_{i+1},\lambda))(g_a(\mu_i,\mathbf{r}_i,\lambda)\sigma_a(\mathbf{r}_i,\lambda)+g_m(\mu_i,\mathbf{r}_i,\lambda)\sigma_m(\mathbf{r}_i,\lambda)))$$

$\mu_i=(\boldsymbol{\omega}_{i-1},\boldsymbol{\omega}_i)$. Then $I_k(\lambda)=\mathbf{E}\sum_{n=0}^{N}Q_n(\mathbf{r}_n,\lambda)\varphi_k^*(\mathbf{r}_n,\lambda)$. Now let divide the atmosphere into n_i layers and assume that in each of them the optical thickness is constant. Assuming that $t=\tau_i(\mathbf{r}_n,\ \mathbf{r}_k,\ \lambda)$, $t_0=\tau_i^{(0)}(\mathbf{r}_n,\ \mathbf{r}_k,\ \lambda)$, $i=0,\ldots,n_i-1$ we can evaluate optical thickness coefficient derivatives. It can be shown to be true that series (8) can be differentiated. It's also can be shown the truthfulness of the estimate: $\left.\frac{\partial I_k}{\partial\tilde{\tau}_i}\right|_{\tilde{\tau}_i=0}=\mathbf{E}\sum_{n=1}^{N}Q_n(\mathbf{r}_n,\tau_i^{(0)},\lambda)\varphi^*(\mathbf{r}_n,\boldsymbol{\omega}_n^*,\tau_i^{(0)},\lambda)\Psi_n(\lambda)$, где $\tilde{\tau}_i=\tau_i-\tau_i^{(0)}$.

Some questions related with solving inverse coefficient problems with respect to full cross-section can be found in [5].

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