CONSTRUCTION OF THE SUBOPTIMAL SOLUTION IN THE BUL PRORAMMING PROBLEM BY ON TWO ESTIMATION OF THE VARIABLES

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I. Let's consider the following Knapsack problem

$$\sum_{j=1}^{n} c_j x_j \to \max \tag{1}$$

$$\sum_{j=1}^{n} a_j x_j \le b \tag{2}$$

$$x_j = 1 \vee 0, \qquad j = \overline{1,n} \tag{3}$$

Without lose of generality assume that the conditions
$c_j > 0,\ \ a_j > 0,\ \ (j = \overline{1,n}),\ b > 0$ and $c_j / a_j \ge c_{j+1}/a_{j+1},\ (j = \overline{1,n-1})$ are satisfied.

Note that the problem (1)-(3) belongs to the NP - integer class i.e. to the class of hard solved problems. To solve this problem there exist some methods as well as "dynamic programming" and "brunching and boundaries" [1-3].

The suboptimal solution of the problem (1)-(3) is found by the following known formula: for each $j = 1,2,...,n$

$$\underline{x}^s_j = \begin{cases} 1, \text{ if } & a_j \le b - \sum_{i=1}^{j-1} a_i x_i^s, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

Then the value of the function (1) indeed is $f^s = \sum_{j=1}^{n} c_j x_j^s$.

Note that when one constructs the suboptimal solution by f the formula (4) to the variables x_j $(j = \overline{1,n})$ take value on one. If we can choose the variables as croups then the obtained solution will not be at least worst than obtained by (4) one.

But since the number of the operations to construct the solution containing k units is of order $c_n^k = \frac{n!}{k!(n-k)!}$, when the number of the variables n is enough large and $k \approx \frac{n}{2}$ to construct this solution one needs expansional order time. This is of course non real computing time. So we use the following criteria for $k = 2$

$$(j_1^*, j_2^*) = \arg\max_{j_1 < j_2} \frac{c_{j_1} + c_{j_2}}{a_{j_1} + a_{j_2}} \tag{5}$$

Here $j_1 = 1,2,...,n-1, \quad j_2 = 2,3,...,n$.

To make clear the sense of the criteria (5) we give the following economical interpretation to the problem (1)-(3): Let n - number of object must be used or not used. If the j-th ($j = \overline{1,n}$) object is used, then it needs expenditures a_j ($j = \overline{1,n}$). In this case the income is c_j ($j=\overline{1,n}$). We have to choose such objects the common expenditures for which be no more than b , but income be maximal. Then it is clear that for some objects j_1 and j_2 the income for the unit expenditures is $(c_{j_1}+c_{j_2})/(a_{j_1}+a_{j_2})$. It is natural first to choose the pair (j_1^*, j_2^*) found by the formula (5).

If the condition $a_{j_1^*}+a_{j_2^*}\le b$ is valid, then taking $x_{j_1}=x_{j_2}=1$, we replace $b_i = b-a_{j_1^*}-a_{j_2^*}$. Otherwise using the criteria $j_* = arq\max_j c_j/a_j$ we estimate the variables on one.

The following theorems are valid.

Theorem 1. Let the sets ω_1 and ω_2 are two different subsets of the set $\{1,2,...,n\}$ and $|\omega_1|=|\omega_2|$. If the condition $\max_{j_1\in\omega_1} j_1 \le \min_{j_2\in\omega_2} j_2$ is satisfied then the relation

$$\sum_{j_1\in\omega_1} c_j \Big/ \sum_{j_1\in\omega_1} a_{j_1} \ge \sum_{j_2\in\omega_2} c_{j_2} \Big/ \sum_{j_2\in\omega_2} a_{j_2} \tag{6}$$

Is true.

Proof. Since $c_j>0,\ a_j>0\ \left(j=\overline{1,n}\right)$ we can write (6) as

$$\sum_{j_1\in\omega_1}\sum_{j_2\in\omega_2} c_i a_j \ge \sum_{j_1\in\omega_1}\sum_{j_2\in\omega_2} c_j a_i\,.$$

From this

$$\sum_{j_1\in\omega_1}\sum_{j_2\in\omega_2}\left(c_{j_1}a_{j_2}-c_{j_2}a_{j_1}\right)\ge 0. \tag{7}$$

As it follows from the problem conditions $c_{j_1}a_{j_2}\ge c_{j_2}a_{j_1}$ when $j_1\le j_2$. According to the theorem conditions $j_1\le j_2$ if $j_1\in\omega_1$ and $j_1\in\omega_2$ .Thus the relation $c_{j_1}a_{j_2}-c_{j_2}a_{j_1}\ge 0$ must be valid for all pairs $(j_1,\ j_2)$, $(j_1\in\omega_1,\ j_2\in\omega_2)$. From this we obtain (7) and then validity of the theorem.

Theorem 2. Suppose

$$\max_{j_1\ne j_2}\left\{(c_{j_1}+c_{j_2})/(a_{j_1}+a_{j_2})\right\}=(c_{j_1^*}+c_{j_2^*})/(a_{j_1^*}+a_{j_2^*})$$

Is found and $j_1^*<j_2^*$. Then $j_1^*=1$.

Proof. Suppose the contrary, i.e. $j_{1*}\ge 2$, $\omega_1=\{1,2\}$ and $\omega_2=\{j_{1*},j_{2*}\}$. As the conditions of the theorem 1 are valid for these sets it is true

$$\frac{c_1+c_2}{a_1+a_2}\ge\frac{c_{j_{1*}}+c_{j_{2*}}}{a_{j_{1*}}+a_{j_{2*}}}\ . \tag{8}$$

If in the relation $\dfrac{c_j}{a_j}\ge\dfrac{c_{j+1}}{a_{j+1}}$ there exits at least one strict inequality till the number $j_1=j_{2*}$ then the inequality (8) will be strictly satisfied. But this is contrary to the theorem 2. From other hand if there is not any strict inequality till the number $j_1=j_{2*}$ then the relation (8) will be satisfied as inequality. Thus theorem 2 is proved.

It immediately follows from the theorem 2 when we choose the variables on two by criteria (5) the number j_1^* in each pair (j_1^*, j_2^*) must be chosen in the known on one choice algorithm. Other words the solution found by the criteria (5) can't be worst than found by the on one choice algorithm. The numerical experiments also demonstrate this statement.

II. Now we consider multidimensional problem

$$\sum_{j=1}^{n} c_j x_j \to \max \tag{9}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad i = \overline{1,m} \tag{10}$$

$$x_j = 1 \vee 0, \qquad j = \overline{1,4} \tag{11}$$

Suppose that $c_j > 0$, $a_{ij} \ge 0$, $b_i > 0$ $(i = \overline{1,m};\ j = \overline{1,n})$

Various methods have been developed for the construction of the suboptimal solution of the problem (9)-(11) [3-6].

In these methods the values of the variables are chosen on one by some criteria. If we find a criteria that allows one to estimate the variables as groups then we can wait that the constructed solution will be better. For this purpose we'll use the following criteria.

$$(j_1^*, j_2^*) = \arg\max_{j_1 < j_2} \frac{c_{j_1} + c_{j_2}}{\max_i (a_{ij_1} + a_{ij_2})} \quad . \tag{12}$$

Here $j_1 = 1,2,...,n-1;$ $j_2 = 2,3,...,n$. In the beginning of construction the suboptimal solution we take $x_j^s = 0,\ j = \overline{1,n}$.

Note that for some objects (j_1, j_2) the income for each unit expenditure will be at least $(c_{j_1} + c_{j_2}) / \max_i (a_{ij_1} + a_{ij_2})$. So first the object found by the criteria (12) must be chosen.

If for each $i\ (i = \overline{1,m})$ the relation $a_{ij_1^*} + a_{ij_2^*} \le b_i$ is valid then may be taken $x_{j_1^*} = x_{j_2^*} = 1$ and $b_i^i = b_i - a_{ij_1^*} - a_{ij_2^*}$ $(i = \overline{1,m})$. When we choose the following pair of numbers by the criteria (12) we need not consider the numbers j_1^* and j_2^* . If at least for one number i the relation $a_{ij_1^*} + a_{ij_2^*} > b_i$ is valid, then the variables we'll be chosen on one by known criteria below [7]

$$j_* = \arg\max_j \frac{c_j}{\max_i a_{ij}} \quad .$$

Theorem 3. The maximal number of the operations required by the method of on two choice of the variables is of order $o(mn^3)$, i.e. has polynomial time complexity.

Note that in the work the software's have been developed for the construction the suboptimal solutions both of the problems (1)-(3) and (9)-(11). The numerical experiments carried out on the different examples. The results of these experiments are given in the tables below.

Table 1.

n	100	500	1000	2000	3000	5000	10000
N	2	3	5	5	6	6	7

Table 2.

$m \times n$	20×100	20×500	20×1000	20×2000	20×3000
N	4	5	5	7	8

The coefficients of the considered problem are integers satisfying the conditions $0 < a_j \le 999$, $0 < c_j \le 999$, $b = [0{,}4 \cdot \sum_{j=1}^{n} a_j]$,

$$0 < a_{ij} \le 999, \quad 0 < c_j \le 999, \qquad b_i = [0{,}4 \cdot \sum_{j=1}^{n} a_{ij}],\ i = \overline{1, m}$$

Ten different 10 dimensional have been solved by known on one choice and offered here on two choice methods. In the tables n denotes the number of variables, m -number of restrictions, N - the number of cases when on two choice method gives better results. In the rest of cases both methods give the same results.

The results of the problem (1)-(3) are given in the table 1, of the problem (9)-(11) in the table 2. As we see the on two choice method gives better results for all problems and increasing of the dimension leads to the increasing of the number of good results.

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