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APPLICATION OF HYBRID METHODS TO THE SOLUTION OF VOLTERRA INTEGRAL EQUATIONS

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As is known, many problems of scientific-technical problems are reduced to the solution of integral equations with variable boundaries. The scientists are engaged in approximate solutions of such equations for a long time. One of the papers in this field belongs to V.Volterra. He is the initiator in investigation and application of integral equations with variable boundary. Therefore, in honour of Volterra, these equations are called Volterra equations. Mainly, different variations of the quadrature method are used in numerical solution of these equations. Here the hybrid type concrete onestep method is applied to numerical solution of Volterra integral equations.

Introduction. Consider the following Volterra integral equation of second kind:

$$y(x) = f(x) + \int_{x_0}^{x} K(x,s,y(s))ds, \quad x \in [x_0, X], \tag{1}$$

where the sufficiently smooth functions $f(x)$, $K(x,s,y)$ are given on the segment $[x_0, X]$ and in the domain $G = \{x_0 \le s \le x \le X, |y| \le a\}$, respectively, that are known. We assume that integral equation (1) has a unique continuous solution determined on the segment $[x_0, X]$. Denote this solution by the function $y(x)$ and find its approximate values. For that we partition the segment $[x_0, X]$ into N equal parts by means of the constant step $0 < h$ and determine the partitioning points in the form: $x_m = x_0 + mh$ $(m = 0,1,2,...,N)$. Approximate values of the function $y(x)$ at the points x_m $(m = 0,1,2,...,N)$ are denoted by y_m, the exact ones by $y(x_m)$.

The classic method of solution to integral equations (1) is the quadrature method that in one variant may be written in the form (see, for example, [1], [2]):

$$y_n = f_n + h\sum_{i=0}^{n} a_i K(x_n, x_i, y_i) \ (n = 0,1,2,...,N), \tag{2}$$

where $f_n = f(x_n)$, a_i $(i = 0,1,...,n)$ are the coefficients of the quadrature formula. As it follows from (2), the number of inversions to calculations of the kernel $K(x,s,y(s))$ increases due to increase of the values of the quantity n. In order to preserve the amount of computational works in solving integral equations (1) at each step, we suggest a multistep method with constant coefficients, of the form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i} y_{n+i}), \tag{3}$$

where the coefficients α_i, $\beta_i^{(j)}$ $(i, j = 0,1,2,...,k)$ are some real numbers, moreover $\alpha_k \ne 0$, that are determined from the homogeneous system of linear algebraic equations.

Usually, in the theory of numerical methods, onestep and multistep methods are investigated. Each of them has its advantage and lack. Taking into account what has been noted, the scientists suggested to construct the methods on the joint of these directions that could preserve the best properties of onestep and multistep methods and called them hybrid methods. Therefore, here we attempt to apply hybrid methods to the numerical solution of equation (1).

1. Construction of the hybrid method for solving Volterra integral equations.

Consider a special case when the function $K(x,s,y(s))$ is independent of the argument x and denote it by $F(s,y) = K(x,s,y)$. Then from equations (1) we can write:

$$y'(x) = f'(x) + F(x, y(s)), \ y(x_0) = f(x_0). \tag{1.1}$$

There exists a great class of hybrid methods for solving problem (1.1). One of them is of the following form (see [5]):

$$y_{n+1} = y_n + f_{n+1} - f_n + h(3F(x_n + h/3, y_{n+1/3}) + F(x_n + h, y_{n+1}))/4, \quad (1.2)$$

This is an onestep method and has degree of accuracy $p = 3$. Formally, method (1.2) may be considered as a twostep one taking into account that three points x_n, $x_n + h/3$, $x_n + h$ participate in it. Since $x_n + h/3$ is not contained in the set of partitioning points, method (1.2) may be considered as an onestep method. However, method (1.2) may be replaced by the following one:

$$y_{n+1} = y_n + h(3f(x_n + h/3,(4y_n + 5y_{n+1})/9 - 2hf(x_n + h, y_{n+1})/9) + f(x_n + h, y_{n+1}))/4, \quad (1.3)$$

This method is an onestep method and has degree of accuracy $p = 3$. Here, on the base of method (1.2) we construct a method of type (3). For that, we show that it is possible to apply method (1.2) to finding the numerical solution of equation (1). To that end, we use some values of the solution of the function $y(x)$ determined as:

$$y(x_{n+1}) = f_{n+1} + \int_{x_0}^{x_{n+1}} K(x_{n+1}, s, y(s))ds, \quad y(x_n) = f_n + \int_{x_0}^{x_n} K(x_n, s, y(s))ds.$$

Consider the difference $y(x_{n+1}) - y(x_n)$, then we have:

$$y(x_{n+1}) - y(x_n) = f_{n+1} - f_n + \int_{x_0}^{x_n} (K(x_{n+1}, s, y(s)) - K(x_n, s, y(s)))ds + \int_{x_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds. \quad (1.4)$$

Using the Lagrange theorem, we can write:

$$K(x_{n+1}, s, y(s)) - K(x_n, s, y(s)) = hK'_x(\xi_n, s, y(s)),$$

where $x_n < \xi_n < x_{n+1}$.

Consequently, we can write:

$$\int_{x_0}^{x_n} (K(x_{n+1}, s, y(s)) - K(x_n, s, y(s)))ds = h\int_{x_0}^{x_n} K'_x(\xi_n, s, y(s))ds.$$

Considering the conditions imposed on the function $K(x, s, y(s))$, we get

$$\left| \int_{x_0}^{x_n} K'_x(x, s, y(s))ds \right| \le M .$$

Then we can write

$$h\int_{x_0}^{x_n} K'_x(\xi_n, s, y(s))ds = O(h). \quad (1.5)$$

If we take into account the ones obtained in (1.4), we can write

$$y_{n+1} - y_n = f_{n+1} - f_n + \int_{x_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds. \quad (1.6)$$

The integral replaced by a small quantity of first order with respect to h may be replaced by a small quantity of higher order with respect to h by adding an intermediate point while investigating the difference of the function $y(x)$, for example in the following way:

$$y(x_{n+1}) - 2y(x_{n+1/2}) + y(x_n).$$

Indeed, in this case we have:

$$y(x_{n+1}) - 2y(x_{n+1/2}) + y(x_n) = f_{n+1} - 2f_{n+1/2} + f_n + \int_{x_0}^{x_n} (K(x_{n+1}, s, y(s)) - 2K(x_{n+1/2}, s, y(s)) + K(x_n, s, y(s)))ds + \int_{x_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds - 2\int_{x_n}^{x_{n+1/2}} K(x_{n+1/2}, s, y(s))ds. \tag{1.7}$$

It is known that

$$K(x_{n+1}, s, y(s)) - 2K(x_{n+1/2}, s, y(s)) + K(x_n, s, y(s)) = \frac{h^2}{4} K''_{x^2}(\eta_n, s, y(s)), \tag{1.8}$$

where $x_n < \eta_n < x_{n+1}$.

Thus, we get that in estimating the first integral participating in (1.4), we apply (1.8) and repeat what has been written above, change the integral by second order quantity of the following form:

$$\int_{x_0}^{x_n} (K(x_{n+1}, s, y(s)) - 2K(x_{n+1/2}, s, y(s)) + K(x_n, s, y(s))) = O(h^2).$$

Notice that while approximating the differential equation $y'(x) = f(x, y)$, at the point x_n, its left hand side, i.e. the derivatives $y'(x_n)$ may be replaced as: $y_{n+1} - y_n = hy'_n + O(h^2)$. A method for its solution may be constructed as follows:

$$y_{n+1} = y_n + h\sum_{i=0}^{k} \beta_i f(x_{n+i}, y_{n+i}).$$

By inspection β_i $(i = 0,1,2,...,k)$, we can construct a method that approximates equation $y'(x) = f(x, y)$ with order p, where $p > 2$. Taking into account this circumstance and using the following change of the integral

$$\int_{x_n}^{x_n+\alpha h} K(x_n + \alpha h, s, y(s))ds \approx h(3K(x_n + \alpha h, x_n + h/3, y(x_n + h/3)) + K(x_n + \alpha h, x_n + h, y(x_n + h)))/4,$$

from (1.6) we have

$$y_{n+1} - y_n = f_{n+1} - f_n + h(3K(x_{n+1}, x_{n+1/3}, y_{n+1/3}) + K(x_{n+1}, x_{n+1}, y_{n+1}))/4. \tag{1.9}$$

The method that we'll construct on the basis of the method (1.2) following by idea of construction the *k*-step method investigated in [5] for solving equation (1), we can write in the form:

$$y_{n+1} - y_n = f_{n+1} - f_n + h(K(x_{n+1/3}, x_{n+1/3}, y_{n+1/3}) - K(x_{n+1/3}, x_{n+1}, y_{n+1}) + 2K(x_{n+1}, x_{n+1/3}, y_{n+1/3}) + 2K(x_{n+1}, x_{n+1}, y_{n+1}))/4. \tag{1.10}$$

For calculating $y_{n+1/3}$ we suggest the following scheme:

$$y_{n+1/3} = (4y_n + 5y_{n+1})/9 + f_{n+1/3} - (4f_n + 9f_{n+1})/9 - 2hK(x_{n+1}, x_{n+1}, y_{n+1})/9. \tag{1.11}$$

After taking into account (1.11) in (1.10) or in (1.9), we get an implicit method for use of which the predictor corrector method is suggested. In one variant, the predictor-corrector method has the following form:

$$\tilde{y}_{n+1} = y_n + f_{n+1} - f_n + hK(x_{n+1}, x_n, y_n), \tag{1.12}$$

$$\hat{y}_{n+1} = y_n + f_{n+1} - f_n + h(K(x_{n+1}, x_n, y_n) + K(x_{n+1}, x_{n+1}, \tilde{y}_{n+1}))/2, \tag{1.13}$$

$$\bar{y}_{n+1/3} = f_{n+1/3} - (4f_n + 9f_{n+1})/9 + (4y_n + 5\hat{y}_{n+1})/9 - 2hK(x_{n+1}, x_{n+1}, \hat{y}_{n+1})/9, \tag{1.14}$$

$$y_{n+1} = y_n + f_{n+1} - f_n + h(3K(x_{n+1}, x_{n+1/3}, \bar{y}_{n+1/3}) + K(x_{n+1}, x_{n+1}, \hat{y}_{n+1}))/4. \tag{1.15}$$

Here, method (1.12) is the Euler method, the relation (1.13) is the trapezoid method. Together, these methods may be called the Runge-Kutta method of second order. Method (1.14) was constructed for calculating the approximate values of quantities $y(x_n + h/3)$, that may be replaced by any other method and accuracy of method (1.15) should be taken into account. Notice that the methods participating in the predictor-corrector method are stable and therefore the convergence of method (1.12)-(1.15) gives rise to doubts (see [6]).

Now construct an algorithm for using the represented predictor-corrector method, remark, that in the next algorithm we'll suggest initial value $y(x_0)$ in the form: $f_0 = f(x_0)$.
To approximate the solution of the Volterra integral equation

$$y(x) = f(x) + \int_{x_0}^{x} K(x,s,y(s))ds, \quad x_0 \le x \le X,$$

at N equally spaces numbers in the interval $[x_0, X]$:

INPUT endpoints x_0, X; initial value f_0; functions $f(x)$ and $K(x,y,z)$; positive integer N.
OUTPUT approximation y_n to $y(x_n)$ at the N values of x.
STEP 1 Set $h = (X - x_0)/N$;
OUTPUT (x, y_0).
STEP 2 For $n = 0,1,2,...,N-1$ do Steps 3-8
STEP 3 Set $x_n = x_0 + nh$
STEP 4 Compute $\tilde{y}_{n+1}$ by formula (1.12).
STEP 5 Compute $\hat{y}_{n+1}$ by formula (1.13).
STEP 6 Compute $\bar{y}_{n+1/3}$ by formula (1.14).
STEP 7 Compute y_{n+1} by formula (1.15).
STEP 8 OUTPUT ($n; \hat{y}_n; y_n$).
STEP 9 STOP.

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