OPTIMIZATION OF THE NUMBER OF CONSTANCY INTERVALS OF PIECEWISE-CONSTANT CONTROL FUNCTIONS WITH UNCERTAIN INFORMATION

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In the paper, optimal control problems for concentrated systems on the class of piecewise constant controls with uncertain information on initial conditions and on parameters of the systems, are investigated. The piecewise constant values of the controls, the boundaries and the number of constancy intervals of the controls are optimized. The formulas for the gradient of the target functional, which allow to use efficient first-order optimization methods for numerical solution of the problem, are obtained, and the algorithm of the numerical solving of optimization problem of constancy intervals number of controls is proposed.

Assume that vector y from compact set $Y \subset E^m$ has a distribution on it defined by distribution function $\wp_Y(y)$ [1], and at each value of the vector y the controlled object is described by the following system of ordinary differential equations:

$$\dot{x}(t) = f(x,u,y), \ t \in (0,T], \qquad x(0) = x_0 \in X_0 \subset E^n, \quad y \in Y. \tag{1}$$

Here, initial vector x_0 takes on values from given compact set X_0 and has a distribution on the set X_0 specified by distribution function $\wp_{X_0}(x_0)$, where $x = x(t) \in E^n$, $t \in [0,T]$, is phase vector, and $u = u(t) \in E^r$ ($t \in [0,T]$) is control. The trajectory $x(t) = x(t;u,y,x_0)$, $t \in [0,T]$, corresponds to every admissible control $u(t)$, to the values of the vector of parameters y, and to the initial vector x_0 on the strength of (1).

Control of system (1) is considered on the class of piecewise constant functions [2-6], which take on constant values on each half-interval $[\tau_{j-1}, \tau_j)$, $j = 1,...,L$, resulting from the partitioning of the interval $[0,T]$ $(L-1)$ by the optimized points τ_j, $j = 1,...,L-1$, that is

$$\begin{aligned} &u(t) = v_j = const, \quad t \in [\tau_{j-1}, \tau_j), \qquad v_j \in E^r, \\ &\tau_{j-1} \le \tau_j, \quad j = 1,...,L, \qquad \tau_0 = 0, \ \ \tau_L = T, \end{aligned} \tag{2}$$

and the values of the control $v_j \in E^r$, $j = 1,...,L$ must belong to some given admissible set U, particularly, to the following parallelepiped:

$$U = \{v : v = (v_1,...,v_L), \ \alpha_j \le v_j \le \beta_j, \ v_j, \alpha_j, \beta_j \in E^r, \ j = 1,...,L\}. \tag{3}$$

The problem consists in finding piecewise constant values of the control $u(t)$, that is, values of the finite-dimensional vectors $v_j \in E^r$, $j = 1,...,L$, the boundaries of their constancy intervals defined by the vector $\tau = (\tau_1,...,\tau_{L-1})$, and also L – the number of constancy intervals of the controls when the given functional

$$J(u) = I(v,\tau) = \int_Y \int_{X_0} \left\{ \int_0^T f^0(x(t;u,x_0,y),u(t))dt + \Phi(x(T;u,x_0,y)) \right\} d\wp_{X_0}(x_0) d\wp_Y(y) \tag{4}$$

takes on its minimal value under conditions (1)-(3). It is supposed that given functions f^0, Φ and the vector-function f together with their partial derivatives of the arguments are continuous.

It is recommended to use finite dimensional first order optimization methods for finding the solution to problem (1)–(4). To this purpose, analytical formulas for the gradient of the target functional $gradJ(u)=\nabla I(v,\tau)\ =(\nabla_v I(v,\tau),\nabla_\tau I(v,\tau))$ are obtained.

Theorem. The gradient of the target functional of problem (1)-(4) in the space of control parameters $(v,\tau)\in E^{2L-1}$ under the fulfilment of the imposed conditions on the functions participating in the problem is defined by following formulas:

$$\frac{dI(v,\tau)}{dv_{ij}}=\int_Y\int_{X_0}\left\{\int_{\tau_{j-1}}^{\tau_j}\left[\frac{\partial f^0(x,u,t)}{\partial u_i}-\frac{\partial f^{\mathrm{T}}(x,u,y)}{\partial u_i}\psi(t;u,x_0,y)\right]dt\right\}d\wp_{X_0}(x_0)d\wp_Y(y), \tag{5}$$

$$j=1,...,L,\quad i=1,...,r.$$

$$\frac{dI(v,\tau)}{d\tau_j}=(v_j-v_{j+1})^{\mathrm{T}}\int_Y\int_{X_0}\left\{\left[\frac{\partial f^0(x,u)}{\partial u}-\frac{\partial f^{\mathrm{T}}(x,u,y)}{\partial u}\psi(t;u,x_0,y)\right]\Bigg|_{t=\tau_j}\right\}d\wp_{X_0}(x_0)d\wp_Y(y), \tag{6}$$

$$j=1,...,L-1.$$

where $\psi=\psi(t)\in E^n$.

Denote by $J_L^*=J^*(v^L,\tau^L,L)$ the minimal value of the functional of problem (1)–(4) at the given number L of the constancy intervals of the controls, and by v^L,τ^L – the optimal piecewise constant control and boundaries of the constancy intervals correspondingly. It is evident that $J_L^*=J^*(v^L,\tau^L,L)$ as the third argument of the function L is non-increasing; i.e., in the general case,

$$J^*(u^*)\le J^*(v^{L_1},\tau^{L_1},L_1)\le J^*(v^{L_2},\tau^{L_2},L_2)\ \text{ at }\ L_1>L_2, \tag{7}$$

where $J^*=J^*(u^*)$ is the optimal value of the functional of initial control problem (1)–(4) under the condition that the control functions are piecewise constant. Thus, from (7), it follows that by increasing the number of constancy intervals, it is possible to reduce only the optimal values of the target functional and to draw it near to J^* as mush as one wants:

$$\lim_{L\to\infty}J^*(v^L,\tau^L,L)=J^*.$$

In case, if the solution to optimal control problem (1)–(4) on the class of piecewise constant functions is a piecewise constant control (relay, non-sliding), it is clear that there is some L^* for which

$$J^*(v^L,\tau^L,L)=J^*(u^*)\quad \text{at}\quad L>L^*.$$

As a rational ("optimal") number of the constancy intervals of the control, we propose to accept such a minimal value $\overline{L}$ at which one of the following inequalities holds true for the first time:

$$\Delta J^*(v^{\overline{L}},\tau^{\overline{L}},\overline{L})=\left|J^*(v^{\overline{L}+\Delta L},\tau^{\overline{L}+\Delta L},\overline{L}+\Delta L)-J^*(v^{\overline{L}},\tau^{\overline{L}},\overline{L})\right|\le\delta, \tag{8}$$

$$\Delta J^*(v^{\overline{L}},\tau^{\overline{L}},\overline{L})\big/J^*(v^{\overline{L}},\tau^{\overline{L}},\overline{L})\le\delta, \tag{9}$$

$$\Delta J^*(v^{\overline{L}},\tau^{\overline{L}},\overline{L})\big/\Delta L\le\delta, \tag{10}$$

where $\Delta L>0$ is given integer defining the increment of the number of constancy intervals of the control, and δ is given positive number defined by the required accuracy for the solution of the problem of the optimization of the number of constancy intervals of the control.

It is possible to use any algorithm of one-dimensional search, for example, bisection method or golden section method, for the determination of the sought rational number $\overline{L}$ of constancy intervals of the control.

According to the results of the solution of the control problem on the class of piecewise constant control actions in the presence of the predetermined number L of constancy intervals, it is possible to consider the issue of reducing the number L, if for optimal values τ^L and v^L on any two consecutive j and $(j+1)$th intervals $(j = 0,...,L-1)$, one of the following conditions holds true:

$$\left|\tau_j^L - \tau_{j+1}^L\right| < \delta_1, \tag{11}$$

$$\left|v_{ij}^L - v_{ij+1}^L\right| \le \delta_2, \quad i = 1,...,r, \tag{12}$$

for any given sufficiently small values $\delta_1, \delta_2 > 0$. In these cases, we can combine the j-th and $(j+1)$th intervals, so the number of constancy intervals is decreased by one.

Using formulas (5), (6), apply the described approach to the following test problem, at the heart of which lies a modeling problem with exactly given values of parameter $y = 1$ and initial conditions $x_0 = (5;0)$, involving known optimal control $v^* = (v_1^*, v_2^*, v_3^*) = (1;-1;1)$, $\tau^* = (\tau_1^*, \tau_2^*) = (0.95;4.55)$:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = yu - \sin x_1, \end{cases} \quad x_1(0) \in [4.8;5.2], \quad x_2(0) = 0, \tag{13}$$
$$y \in [0.9;1.1], \quad |u(t)| \le 1, \quad t \in [0;5],$$

$$J(u) = \int_{4.8}^{5.2}\int_{0.9}^{1.1}\left[x_1^2(5;x_0,y) + x_2^2(5;x_0,y)\right] d\wp_{X_0}(x_0) d\wp_Y(y) \to \min_u . \tag{14}$$

Distribution functions $\wp_{X_0}(x_0)$ and $\wp_Y(y)$, we choose as

$$\wp_{X_0}(x_0) = \frac{1}{0.4}(x_0 - 4.8), \quad \wp_Y(y) = \frac{1}{0.2}(y - 0.9),$$

Then, functional (14) takes on the form

$$J(u) = \frac{1}{0.08}\int_{4.8}^{5.2}\int_{0.9}^{1.1}\left[x_1^2(5;x_0,y) + x_2^2(5;x_0,y)\right] dx_0 dy \to \min_u . \tag{15}$$

We suppose that the number of constancy intervals of the piecewise constant control $u(t)$ equals three, i.e. $L = 3$. Thus, the vector $(\tau, v) = (\tau_1, \tau_2, v_1, v_2, v_3)$ is optimized.

The results of the numerical experiments for the solution of problem (13), (15) are given in table 1 at various initial values (τ^0, v^0) of the control vector (τ, v), where the precision of the optimization is $\varepsilon = 0.001$.

Table 1. Numerical results of the solution of the problem.

N	(τ^0, v^0)	$I(\tau^0, v^0)$	(τ^*, v^*)	$I(\tau^*, v^*)$	Number of iterations
1	(0.78;3.46; 0.70;-0.60;0.50)	33.0325	(0.9651; 4.5500; 1.0000; -1.0000; 1.0000)	13.0587	34
2	(0.78;3.46; 2.00;-2.00;0.50)	7.9791	(0.9709; 4.5500; 1.0000; -1.0000; 1.0000)	13.0591	28
3	(0.52;2.73; 0.80;-0.80;0.40)	44.2627	(0.9659; 4.5485; 1.0000; -1.0000; 1.0000)	13.0595	68
4	(0.28;3.26; 0.26;-0.40;0.32)	43.1466	(0.9690; 4.5500; 1.0000; -1.0000; 1.0000)	13.0589	35
5	(0.64;2.82; 0.85;-0.30;0.40)	43.1901	(0.9668;4.5500; 1.0000; -1.0000; 1.0000)	13.0588	27

The results of the optimization of the piecewise constant controls in problem (13), (15) with five intervals of control constancy for two distinct initial points of the iterative process are given in table 2. Using conditions (11), (12), from the obtained results of the optimization given in table 2, it follows that the optimal solution obtained from the first initial point has the components τ_1^*, τ_2^* of the vector τ^* that satisfy condition (11), and the components v_3^*, v_4^* of the vector v^* satisfying condition (12). The values of the components τ_1^*, τ_2^*, and τ_3^* of the vector τ^* satisfy condition (11) for the optimal solution obtained from the second initial point. Therefore, combining the components with close values, we obtain the results similar to those given in table 1, and also $L^* = 3$.

Table 2. Numerical results of the solution of the problem when $L = 5$.

N	(τ^0, v^0)	$I(\tau^0, v^0)$	(τ^*, v^*)	$I(\tau^*, v^*)$	Number of iterations
1	(0.62;0.78;2.26; 3.76;0.4;0.5; -0.6;-0.8;0.8)	31.1464	(0.950;0.971;1.875; 4.552;1.00;0.613; -1.00;-1.00;1.00)	13.0589	16
2	(1.11;0.76;0.81; 4.76;0.45;0.85; -0.7;-0.8; 0.8)	19.1692	(0.983;0.983;0.983; 4.550;1.00;0.845; -0.646;-1.00;1.00)	13.0774	5

References

1. Gabasov R., Kirillova F.M. Maximum principle in optimal control theory. (In Russian) Nauka i Tekhnika, Moscow (1974) 272 p.
2. Rahimov A.B. Numerical solution of optimal control problems on the class of piecewise constant functions with uncertain information. (In Russian) "Information technologies and mathematical modeling" (ITMM-2009): Proceedings of VIII All-Russian Scientific-and-Practical Conference with international participation (November 13-14, 2009, Kemerovo area, Anzhero Sudzhensk, Russia)-Tomsk: Publishing house of Tomsk University, 2009, Part 2, pp. 266-271.
3. Ismibeyli R.E., Rahimov A.B. Control on the class of piecewise constant functions with uncertain information on its initial state and on parameters. (In Russian) Proceedings of the International Scientific-and-Practical Conference on "Actual problems of mathematics, informatics, mechanics, and control theory" devoted to the 60th anniversary of prof. Biyarov T.N., November 19-20, 2009, Almaty, Kazakhstan, Part 2, pp. 376-380.
4. Rahimov A.B. Control of a system on the class of piecewise constant functions involving incomplete information on initial conditions and on parameters. (In Russian) Transactions of the National Academy of Sciences of Azerbaijan, Series of Physical-and-Mathematical and Technical Sciences, vol. XXIX, 2009, № 6, pp. 65-70.
5. Aida-zade K.R., Rahimov A.B. Solution of optimal control problem in class of piecewise-constant functions. Automatic Control and Computer Science, Allerton Press, Inc., New York, 2007, Vol. 41, No. 1, pp. 18-24.
6. Aida-zade K.R., Rahimov A.B. Numerical solving of control problem for object with uncertain information on its initial state and parameters. Automatic Control and Computer Science, Allerton Press, Inc., New York, 2010, Vol. 44, No. 2, pp. 61-68.