FEEDBACK CONTROL AT OBSERVED POINTS

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Let homogenous rods of the length l be sequentially (or simultaneously, but independently of each other) heated in a heating stove at the expense of the temperature $\vartheta(t)$ produced by an external source and identical in all the heating stove. Then, the process of heating each rod is described by the following differential equation of parabolic type:

$$u_t(x,t) = a^2 u_{xx}(x,t) + \alpha\left[\vartheta(t) - u(x,t)\right], \ (x,t) \in \Omega = (0,l)\times[0,T], \tag{1}$$

with boundary conditions

$$u_x(0,t) = \lambda\left[u(0,t) - \vartheta(t)\right], \quad t \in (0,T], \tag{2}$$

$$u_x(l,t) = -\lambda\left[u(l,t) - \vartheta(t)\right], \quad t \in (0,T], \tag{3}$$

where $a^2 = \dfrac{k}{c\rho} = const > 0$ is thermal conductivity coefficient; $\alpha = \dfrac{h}{c\rho}$ and $\lambda = \dfrac{h}{k}$ are reduced coefficients of heat exchange between environment and the rod in the heating stove along the length and at the ends of the rod correspondingly; h is heat exchange coefficient; k is heat conductivity coefficient; c is specific heat coefficient; ρ is the density of the material.

The initial temperature of the rods, for the sake of simplicity, is considered constant along their lengths, but different for each rod. At that, we have some admissible set (interval) $B = [\underline{B}, \overline{B}]$ of possible values of the temperature:

$$u(x,0) = b = const \in B, \ x \in [0,l], \tag{4}$$

and the density function $\rho_B(b)$ of initial temperatures is given, where

$$\int_B \rho_B(b)db = 1, \quad \rho_B(b) \ge 0, \quad b \in B. \tag{5}$$

The current temperature $u(x_i,t)$, $i = 1,2...,L$ is measured at L points $x_i \in [0,l]$ of all the rods with the help of sensors. Depending on the values of the temperature at the sources, the current temperature $\vartheta(t)$ is assigned inside the heating stove.

Let $\gamma_i, i = 1,2,...,L$ be weighting coefficients characterizing the importance of taking into account the values of the temperature at the measured points, at that

$$\sum_{i=1}^{L} \gamma_i = 1, \ 0 \le \gamma_i \le 1, \ i = 1,2,...,L\,. \tag{6}$$

The value

$$\tilde{u}(t) = \sum_{i=1}^{L} \gamma_i u(\bar{x}_i, t), \ t \in [0,T]$$

is the current value of the "averaged" temperature of the rod according to the measured data. This value is used to form a feedback control for the heating stove:

$$\vartheta(t) = \vartheta(t; K, \bar{\gamma}) = K(t)\tilde{u}(t) = K(t)\sum_{i=1}^{L} \gamma_i u(\bar{x}_i, t), \tag{7}$$

where $K(t)$ is control parameter defining the temperature of the heating stove. The vector $\gamma = (\gamma_1, \gamma_2, ..., \gamma_L)$, in the general case, may be a function of time, but for the sake of simplicity, we consider it to be invariable and unknown.

Taking into account (7) in (1)-(3), we obtain the boundary problem of the form:

$$u_t(x,t) = a^2 u_{xx}(x,t) + \alpha \left[K(t) \sum_{i=1}^{L} \gamma_i u(\bar{x}_i, t) - u(x,t) \right], \ (x,t) \in \Omega = (0,l) \times [0,T], \tag{8}$$

$$u_x(0,t) = \lambda \left[u(0,t) - K(t) \sum_{i=1}^{L} \gamma_i u(\bar{x}_i, t) \right], \quad t \in (0,T], \tag{9}$$

$$u_x(l,t) = -\lambda \left[u(l,t) - K(t) \sum_{i=1}^{L} \gamma_i u(\bar{x}_i, t) \right], \quad t \in (0,T] \ . \tag{10}$$

Problem (8)-(10) is called a pointwise loaded problem, as unknown values of the phase variable at different points of the space variable are present in its right-hand side [1].

In practical applications, there may be certain technological constraints imposed on the control parameter $K(t)$:

$$\underline{K} \le K(t) \le \overline{K} \,, \qquad t \in [0,T], \tag{11}$$

where $\underline{K}$, $\overline{K}$ are given upper and lower admissible values of the magnification constant correspondingly.

Suppose we have the following performance criterion:

$$J(K,\bar{\gamma}) = \int_B I(K,\gamma;b) \rho_B(b) db + \varepsilon_1 \| K(t) - K_0 \|^2_{L_2[0,T]} + \varepsilon_2 \| \gamma - \gamma_0 \|^2_{R^L} \,, \tag{12}$$

$$I(K,\gamma;b) = \int_0^l \mu(x) [u(x,T;K,\gamma,b) - U(x)]^2 dx \,, \tag{13}$$

where $U(x)$ is given function; $\mu(x) \ge 0$ is given weighting function; $u(x,t;K,\gamma;b)$ is the solution to boundary problem (8)-(10) in the presence of control parameters $K = K(t), \gamma$ and of initial condition $u(x,0) = b, x \in [0,l]$; $\varepsilon_1 > 0, \varepsilon_2 > 0, K_0 \in R^1, \gamma_0 \in R^L$ are regularization parameters satisfying (6) and (11).

The case when the observation over the heating process at the points $\bar{x}_i \in [0,l],\ i = 1,2...,L_x$ of the rod is carried out not continuously, but at given discrete moments of time $\bar{t}_j \in [0,T],\ j = 0,1,...,L_t,\ \ t_0 = 0,\ \bar{t}_{L_t} = T$, is of practical interest. The temperature inside the heating stove is assigned according to the results of observation, and is constant at the interval of time between any two observations, and is determined, for example, by the formula:

$$\vartheta(t) = K_j \sum_{i=1}^{L_x} \gamma_i u(\bar{x}_i, \bar{t}_{j-1}) = const,\ K_j = const,\ t \in [\bar{t}_{j-1}, \bar{t}_j)\ \ j = 1,2,...,L_t. \tag{14}$$

It is possible to make use of the "memory" to measure the values of the temperature at time using the formula

$$\vartheta(t) = K_j \sum_{i=1}^{L_x} \sum_{\nu=1}^{j-1} \gamma_{ij-1} u(\bar{x}_i, \bar{t}_{j-1}) = const, \quad t \in [\bar{t}_{j-1}, \bar{t}_j), \tag{15}$$

where $\gamma_{i\nu}$ are weighting coefficients of the importance of taking into account the value of the temperature at i^{th} point $\bar{x}_i$ at ν^{th} measurement at $(j-1)^{\text{th}}$ time interval, i.e. at moments of time $\bar{t}_\nu,\ \nu = 0,...,j-1$.

The control problem is reduced to seeking the finite-dimensional vector of parameters $K = (K_1,...,K_{L_t}),\ \gamma = (\gamma_1,...,\gamma_{L_x})$ in case of (14), and the matrix $\gamma = ((\gamma_{ij})),\ i = 1,...,L_x,\ j = 1,...,L_t$ in case of (15). For both cases, the computation given below is not altered significantly; that is why we consider only control of type (7).

For the numerical solution to parametrical optimal control problem (8)-(13), i.e. for the determination of function $k(t)$ and of finite-dimensional vector of parameters γ, we propose to use first order optimization methods.

From (8)-(13), taking into account the independence of the initial conditions of each other, and therefore the independence of the solution to boundary problems (8)-(10) for different initial conditions $u(x,0) = \mathrm{b} \in \mathrm{B}$, it follows the validity of the formula:

$$\begin{pmatrix} grad_K J(K,\gamma) \\ grad_\gamma J(K,\gamma) \end{pmatrix} = \begin{pmatrix} \int_B grad_K I(K,\gamma;b)\rho_B(b)db \\ \int_B grad_\gamma I(K,\gamma;b)\rho_B(b)db \end{pmatrix}.$$

That is why, in order to apply first order optimization methods, obtain formulas for the gradient of functional (13) taking into account boundary problem (8)-(10) involving any admissible initial condition:

$$u(x,0) = b,\ x \in [0,l],\ b \in B. \tag{16}$$

When solving problem (8)-(13) numerically with the application of standard first order optimization procedures, at each step of the iteration procedure, we use the gradient of the functional. With that end in view, at the current control, it is necessary to solve loaded boundary problem (8)-(10) and the following conjugate integral-and-differential equation:

$$\psi_t(x,t) = -a^2\psi_{xx}(x,t) - \alpha\left[K(t)\int_0^l \psi(\xi,t)d\xi \sum_{i=1}^{L}\gamma_i\,\delta(x-\bar{x}_i) - \psi(x,t)\right], \tag{17}$$
$$x \in (\bar{x}_{i-1},\bar{x}_i),\ t \in [0,T], i = 1,2,..,L\ ,$$

with boundary and initial conditions

$$\psi(x,T) = 2\mu(x)[u(x,T) - U(x)],\ \ x \in [0,l], \tag{18}$$

$$\psi_x(0,t) = \lambda\psi(0,t),\quad x \in [0,T],\ \psi_x(l,t) = -\lambda\psi(l,t),\quad x \in [0,T], \tag{19}$$

and non-local jump condition at intermediate points $\bar{x}_i$, $i = 1,2,...,L$ of observation

$$\psi^+(\bar{x}_i,t) = \psi^-(\bar{x}_i,t),\ i = 1,...,L,$$
$$\psi_x^+(\bar{x}_i,t) - \psi_x^-(\bar{x}_i,t) = -\lambda K(t)\gamma_i(\psi(l,t) + \psi(0,t)),\ i = 1,...,L. \tag{20}$$

Theorem 1. The gradient of the functional in problem (8)-(10) for admissible control parameters $K = K(t), \gamma$ is determined by the following formulas

$$grad_K J(K,\gamma) = \int_B \left[\alpha\int_0^l \psi(x,t)dx \sum_{i=1}^{L}\gamma_i u(\bar{x}_i,t) - \right.$$
$$\left. - a^2\lambda\sum_{i=1}^{L}\gamma_i u(\bar{x}_i,t)\big(\psi(0,t)+\psi(l,t)\big)\right]\rho_B(b)db + 2\varepsilon_1(K(t)-K_0),\ t \in [0,T], \tag{21}$$

$$grad_\gamma J(K,\gamma) = \int_B \left[\int_0^T \left[K(t)u(\bar{x},t)\left(\alpha\int_0^l \psi(x,t)dx - \lambda a^2(\psi(0,t)+\psi(l,t))\right)\right]dt\right]\rho_B(b)db +$$
$$+ 2\varepsilon_2(\gamma-\gamma_0), \tag{22}$$

where $u(x,t) = u(x,t;K,\gamma;b), \psi(x,t) = \psi(x,t;K,\gamma;b)$ are the solutions to the direct and conjugate boundary problems (8)-(13) and (17)-(20) correspondingly at given initial admissible condition $u(x,0) = b$.

Formulas (17)-(20) for the gradient of the functional of problem (8)-(13) can be obtained using method of lines over time for reducing the initial problem to an optimal control problem with respect to a system of ordinary loaded differential equations involving non-local boundary conditions [4]. Then, applying necessary optimality conditions obtained in the work [5] to these problems, and passing to limit when the step of discretization over time tends to zero, we can obtain formulas (17)-(20). Below, we propose to use method of lines to numerically implement the iteration method of gradient projection, namely, to solve the boundary problems: direct (8)-(11) and conjugate (17)-(20). To solve the optimal control problem for the loaded system of differential equations involving non-local boundary conditions, we use the numerical method proposed in the works [2, 3].

In the domain Ω, introduce the lines $t_s = sh_t$, $s = 0,...,N_t$, $h_t = T/N_t$ and notations $u_s(x) = u(x, sh_t)$, $K_s = K(sh_t), s = 0,1,...,N_t$.

Approximate boundary problem (8)-(10) by a boundary problem with respect to the following loaded system of N_t ordinary differential equations involving non-local boundary conditions:

$$a^2 u_s''(x) - (\frac{1}{h_t} + \alpha) u_s(x) + \frac{1}{h_t} u_{s-1} + \alpha K_s \sum_{i=1}^{L} \gamma_i u_s(\bar{x}_i) = 0, \tag{23}$$

$$u_s'(0) = \lambda \left[u_s(0) - K_s \sum_{i=1}^{L} \gamma_i u_s(\bar{x}_i) \right], \quad u_s'(l) = -\lambda \left[u_s(l) - K_s \sum_{i=1}^{L} \gamma_i u_s(\bar{x}_i) \right], \; s = 1,...,N_t, \tag{24}$$

$$u_0(x) = b \in B, \; x \in [0, l]. \tag{25}$$

Target functional (13) is approximated, for example, by the formula

$$I(K, \gamma; b) = \int_0^l \mu(x) \left[u_{N_t}(x) - U(x) \right]^2 dx + \varepsilon_1 h_t \sum_{s=0}^{N_t - 1} (K_s - K_0)^2 + \varepsilon_2 \sum_{i=1}^{L} (\gamma_i - \gamma_{oi})^2 \tag{26}$$

The obtained optimal control problem lies in determining $(N_t + L)$ dimensional vector of parameters $(K, \gamma) = (K_1, ..., K_{N_t}, \gamma_1, ..., \gamma_L)$. In order to solve this problem using gradient projection method, give formulas of the gradient of functional (26):

$$grad\, I(K, \gamma; b) = \left(\frac{\partial I}{\partial K_1}, ..., \frac{\partial I}{\partial K_{N_t}}, \frac{\partial I}{\partial \gamma_1}, ..., \frac{\partial I}{\partial \gamma_L} \right).$$

Conjugate boundary problem (17)-(20) is also approximated with the application of method of lines by loaded second order ordinary differential equations involving non-local boundary conditions

$$a^2 \psi_s''(x) - (\frac{1}{h_t} - \alpha) \psi_s(x) + \frac{1}{h_t} \psi_{s+1}(x) - \alpha K_s \int_0^l \psi_s(x) dx \sum_{i=1}^{L} \gamma_i \delta(x - \bar{x}_i) = 0, \tag{27}$$

$$\psi_s'(0) = \lambda \psi_s(0), \qquad \psi_s'(l) = -\lambda \psi_s(l), \tag{28}$$

$$\psi_s'^{+}(\bar{x}_i) - \psi_s'^{-}(\bar{x}_i) = -\lambda K_s (\psi_s(l) + \psi_s(0)), \; i = 1,...,L, \tag{29}$$

which are solved successively from $s = N_t - 1$ to $s = 1$ provided that

$$\psi_{N_t}(x) = 2\mu(x) \left[u_{N_t}(x) - U(x) \right], \quad x \in (0, l), \tag{30}$$

Then, the components of the gradient of the functional of problem (23)-(26) are determined by the approximation of formulas (21) and (22) in the following way:

$$\frac{dJ}{dK_s} = h_t \sum_{i=1}^{L} \gamma_i \int_B \left[u_s(\bar{x}_i) \left[\alpha \int_0^l \psi_s(x) dx - \lambda a^2 \left(\psi_s(0) + \psi_s(l) \right) \right] \right] db + 2\varepsilon_1 (K_s - K_0), \; s = 1,.., N_t \tag{31}$$

$$\frac{dJ}{d\gamma_i} = h_t \sum_{s=1}^{N_\tau} K_s \int_B u_s(\bar{x}_i) \left[\alpha \int_0^l \psi_s(x)dx - \lambda a^2 \left(\psi_s(0) + \psi_s(l)\right) \right] db + 2\varepsilon_1 (\gamma - \gamma_0), \quad i = 1,...,L. \tag{32}$$

The other specific character of these boundary problems is pointwise loading of equations (23) and integral loading of equations (27), as well as the presence of non-local boundary conditions (24) and (29). In the work [3], for the solution to such problems, a numerical solution method is proposed. It is based on the shift of boundary conditions, for example, from left to tight successively from point $x = 0$ to points $\bar{x}_1,....,\bar{x}_L, x = l$, and as a result we obtain $(L+1)n$ (where n is the order of the system) algebraic equations with respect to $(u_s(\bar{x}_1),...,u_s(\bar{x}_L),u_s(l))$. After solving this system, the initial boundary problem is reduced to a Cauchy problem that is solved from right to left. Analogical approach is proposed in [4] for integrally loaded ordinary differential equations involving non-local boundary conditions.

Note that the statement of the optimal feedback control problem and the approach to its solution given above can be extended onto other classes of optimal control problem with respect to systems with distributed parameters, described by other types of partially differential equations.

The results of the numerical experiments carried out for the solution to optimal control problems (8)–(13) are given in the work.

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