OPTIMAL IMPULSIVE CONTROL IN DISTRIBUTED SYSTEMS

Yegana Ashrafova

Cybernetics Institute of ANAS, Baku, Azerbaijan
y_aspirant@yahoo.com

Nowadays the optimal control theory on special classes of functions represents a developing section of optimization of dynamical systems [1-6]. One of the basic investigation phases of the optimization of systems is the problem of obtaining constructive conditions of the optimality, considering the specific character of this class of systems and allowing to use them for the numerical solution of the problems.

The same problems of optimal control on the class of impulsive functions are considered in [4] for ordinary differential equations. In [5] the problem of placing of oil wells and control by output of oil is considered.

The solution to optimal control problems on the classes of functions that are technically easily implemented is of important value. The optimal control problem when the controls belong to the class of impulsive functions is considered in the given article. Constructive analytical formulas for the gradient of a functional in the optimization problems for the power and the influence span of impulse controls, as well as for coordinates of concentrated sources are investigated. These formulas allow to use the first order optimization methods for solving the problems of optimal control [7]. The results of numerical experiments are also given.

Let us consider the next problem of optimal impulsive control by system with distributed parameters, which consist of the minimization of the next functional

$$J(w)=\alpha_1\int_\Omega [u(x,T;w)-U(x)]^2dx+\alpha_2\left\|v(t)-v^0(t)\right\|^2_{L_2[0,T]}+\alpha_3\sum_{i=1}^{L}\left\|\xi^i-\xi^{i0}\right\|^2_{R^n} \quad (1)$$

with conditions that, the position of controlling object is described by the boundary problem according to parabolic type:

$$u_t=div\big(\sigma(x)grad\,u(x,t)\big)+\sum_{i=1}^{L}v_i(t)b_i(x,t)\delta(x-\xi^i),\ \ x\in\Omega\subset R^n,\quad 0<t\le T, \quad (2)$$

$$u(x,0)=\varphi(x),\quad x\in\Omega, \quad (3)$$

$$u(x,t)\,|_{x\in\Gamma}=\mu(x,t),\, 0<t\le T,\quad \Gamma=\partial\Omega. \quad (4)$$

Here $u=u(x,t)=u(x,t;v)$ – is a phase position of the object, which is determining from the solution of the boundary problem of (2)–(4) on corresponding admissible value of optimized control vector $w=(v(t),\xi)$, with power of sources $v(t)=(v_1(t),....,v_L(t))$, placing on coordinates $\xi^i=(\xi_1^i,...,\xi_n^i)\in\Omega, i=1,...,L$; $v^0(t)=(v_1^0(t),....,v_L^0(t))$ -is a given vector-function, $\xi^{i0}=(\xi_1^{i0},...,\xi_n^{i0})\in\Omega, i=1,...,L$ -is a given vector; R^n – is n-dimensional Euclidian space; L – is a given number of control influences (sources); $\varphi(x),\mu_1(t),\mu_2(t),b_1(x,t),...,b_L(x,t)$, $\sigma(x),U(x),\alpha_i>0, i=1,2,3$, $l>0, T>0$ – are given functions and values, determining the investigated process and the criterions of control on it; $\delta(x)=\delta(x_1)\cdot\delta(x_2)\cdot...\cdot\delta(x_n)$, $\delta(x_i)-$ the generalized Dirac function.

Control influences

$$v_i(t)=\sum_{j=1}^{m_i}q_{ij}\delta(t-\theta_{ij}),\ M=\sum_{i=1}^{L}m_i,\ i=1,..,L, \quad (5)$$

are determined by finite-dimensional vector $v=(q,\theta)\in R^{2LM}$, where q_{ij} is a value of impulsive power of $i-$th source at the moment of θ_{ij}, $j=1,..,m_i$, $i=1,..,L$; m_i $-$is a given number of impulsive influences of $i-$th source, i.e.,

$$v=(q_{11},q_{12},...,q_{1m_1},...,q_{Lm_L},\theta_{11},\theta_{12},...,\theta_{1m_1},...,\theta_{Lm_L}). \tag{6}$$

Let us consider the following constraints on control parameters:

$$\sum_{j=1}^{L}\sum_{j=1}^{m_i}q_{ij}^2\le Q,\ \underline{q_i}\le q_{ij}\le\overline{q_i},\quad 0\le\zeta<\theta_{ij}-\theta_{i-1j}\le\eta,$$

$$\theta_{ij}\in[0,T],\ j=1,...,m_i,i=1,...,L, \tag{7}$$

where $Q,\underline{q_j},\overline{q_j},\zeta,\eta$ are given.

It is considered that, the functions and parameters in the problem of (1)-(7) satisfy all conditions of existence and uniqueness of the boundary problem solution.

The problem consists in optimisation as functionality of sources defining by vector $v=(q,\theta)$, as well as the coordinate $\xi^1,...,\xi^L$ of placing the sources on the domain of Ω

The problem (1)-(7) from one side may be related to the class of parametrical optimal control problems, on the other hand they are problems of finite-dimensional optimization. The considering optimal control problems are equivalent to the problems of the optimization of functional $J(v)$ in the closed admissible domain; so the set of optimal solutions is nonempty.

The controls may be discontinuous in the considering problems, so there isn't a classical solution of these problems. It is known that on the even admissible control there is unique generalized solution of boundary problem (2)-(4) [6].

In the aim of applying first order optimization methods for the determination of optimal control vector, we will obtain analytical formulas for the gradient of functional (1) of the problem investigated. To obtain these formulas by using the method of functional variation of Lagrange the next theorem have been proved.

Theorem With conditions on controls above, the components of the gradient of the functional on parameters of the impulse control influences and on coordinates of source placing in problem of (1)–(7) are determined by the formulas:

$$\frac{dJ(w)}{dq_{ij}}=-\psi(\xi^i,\theta_{ij})b_i(\xi^i,\theta_{ij})+2\alpha_2(q_{ij}-q_{ij}^0),\ i=1,...L,$$

$$\frac{dJ(w)}{d\theta_{ij}}=q_{ij}(\psi(\xi^i,\theta_{ij})b_i(\xi^i,\theta_{ij}))'_t,\ j=1,...,m_i,i=1,...,L,$$

$$\frac{dJ(w)}{d\xi_j^i}=\int_0^T(\psi(\xi_j^i,t)b_i(\xi_j^i,t))'_{x_j}v_i(t)dt+2\alpha_3(\xi_j^i-\xi_j^{i0}),\ j=1,..,n,\ i=1,...,L,$$

where $\psi=\psi(x,t)$ is a solution of the next adjoint problem:

$$\psi_t+div\sigma(x)grad\psi=0,\ x\in\Omega,0<t\le T,$$

$$\psi(x,T)=2\alpha_1(u(x,T)-U(x)),\ x\in\Omega,$$

$$\psi(x,t)\big|_{x\in\Gamma}=0,\ 0<t\le T.$$

Using the obtained formulas for the gradient of the functional, let us apply their results to the next model problems.

Problem 1

Let us consider a problem of heating of the stick by impulse influence, when $L=1$, i.e., there is unique impulse influence on the process. The optimized parameters are the power, span time and the coordinate of source of impulse influence: $v=(q,\theta,\xi)$.

$$u_t = u_{xx} + (x+t)q\delta(x-\xi)\delta(t-\theta), \quad 0 < x < 1, 0 < t \le 1,$$
$$u(x,0) = e^x, \quad 0 \le x \le 1,$$
$$u(0,t) = t+1, \; u(1,t) = e^{t+1}, \quad 0 < t \le 1,$$
$$0 \le \xi \le 1, \; 0 \le \theta \le 1, 0 < q \le 10,$$
$$J(v) = \int_0^1 [u(x,1)-4]^2 dx + 0{,}1(q-3)^2 + 0{,}1(\xi - 0{,}5)^2 + 0{,}1(\theta - 0{,}3)^2 \to \min .$$

The exact value of optimized vector is unknown. The problem has been solved numerically by using the formulas obtained above. The comparative values of the gradients of the functional which is calculated by applying the formulas above, and by using central approximation scheme of derivatives at various values of control vector (q_0, ξ_0, θ_0) are given in table 1.

Table 1.

The calculated values of gradient of the functional

№	(q_0,ξ_0,θ_0)	$(I'_q, I'_\xi, I'_\theta)$ By approximation scheme	$(I'_q, I'_\xi, I'_\theta)$ By obtained formulas
1	(3;0,6;0,1)	(0,400;-0,0364;-0,010)	(0,400;-0,0364;-0,009)
2	(6;0,2;0,2)	(0,600;-0,0559;-0,012)	(0,600;-0,0559;-0,0126)
3	(1;0,2;0,4)	(-0,398;-0,054;0,0312)	(-0,399;-0,054;0,0312)

The results of numerical experiments by using the method of projection of interfaced gradients are given in table 2, at various initial values of control parameters $v_0 = (q_0, \theta_0, \xi_0)$ with the precision of the optimisation $\varepsilon = 0{,}001$. Approximation of the boundary problem was made by using the implicit scheme of grid method with error $O(h_x^2 + h_t)$ including boundary conditions, where h_x, h_t are grid steps correspondingly on variables x and t, $h_x = 0{,}01,\; h_t = 0{,}01$.

Table 2

The numerical results of the problem 1

№	(q_0,ξ_0,θ_0)	(q_*,ξ_*,θ_*)	J_0	J_*	Number of iterations
1	(3;0,6;0,1)	(2,995;0,479;0,219)	2,973	2,568	9
2	(6;0,2;0,2)	(2,998;0,490;0,229)	3,477	2,568	4
3	(1;0,2;0,4)	(3,00;0,494;0,219)	2,978	2,568	3

Problem 2

$$u_t = u_{xx} + (x^2 + t^2)\sum_{i=1}^{2} q_i \delta(x-\xi_i)\delta(t-\theta_i), 0 < x < 1, 0 < t \le 1,$$
$$u(x,0) = e^x, 0 \le x \le 1,$$
$$u(0,t) = t+1, \; u(1,t) = e^{t+1}, 0 < t \le 1, 0 \le \xi_i \le 1, \; 0 \le \theta_i \le 1, \; 0 < q_i \le 10, i = 1,2,$$
$$J(v) = \int_0^1 [u(x,1)-4]^2 dx + 0{,}1((q_1-3)^2 + (q_2-4)^2 +$$
$$+ 0{,}1((\xi_1 - 0{,}5)^2 + (\xi_2 - 0{,}8)^2) + 0{,}1((\theta_1 - 0{,}3)^2 + (\theta_2 - 0{,}5)^2) \to \min.$$

$L = 2$ in this problem. The exact value of optimized vector is unknown. The problem has been solved numerically by using the formulas obtained above. The results of numerical experiments by using the method of projection of interfaced gradients are given in table 3, at various initial values of control vector $v_0 = (q_0, \theta_0, \xi_0)$ with the precision of the optimisation $\varepsilon = 0{,}001$. Approximation of the boundary problem was made as in the previously problem.

Table 3

The numerical results of the problem 2

№	(q_0, ξ_0, θ_0)	(q_*, ξ_*, θ_*)	J_0	J_*	Number of iterations
1	(1;0,2;0,4) (3;0,2;0,3)	(3,026;0,491;0,258) (4,008;0,860;0,299)	3,1177	2,5735	11
2	(2;0,4;0,74) (6;0,6;0,6)	(3,00;0,486;0,260) (3,985;0,888;0,369)	3,3466	2,5727	12
3	(2;0,3;0,2) (4;0,5;0,3)	(2,998; 0,488; 0,259) (3,99;0,911;0,359)	2,68673	2,5729	36
4	(1;0.5;0,2) (2;0,5;0,1)	(2,998;0,488;0,259) (3,996;0,910;0,367)	3,3933	2,5728	15

Analogical researches can be carried out on the processes described by other types of differential equations with private derivatives.

According to simplicity of realization of the impulsive control and their extensive using in technique, the suggested approach to constructing the strategy of control by the objects with concentrated parameters can find the wide application in the systems of control by those objects.

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