SECOND ORDER NECESSARY OPTIMALITY CONDITIONS IN THE DISCRETE TWO-PARAMETER STEP CONTROL PROBLEMS

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Let the controlled system be described by the following discrete two-parametric system of equations

$$z_i(t+1,x+1)= f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x)),\ \ (t,x)\in D_i,\ i=\overline{1,3},\quad (1)$$

with boundary conditions

$$\begin{gathered} z_1(t_0,x)=\alpha_1(x),\quad x=x_0,x_0+1,\dots,X,\quad z_1(t,x_0)=\beta_1(t),\quad t=t_0,t_0+1,\dots,t_1,\\ z_2(t_1,x)=z_1(t_1,x),\quad x=x_0,x_0+1,\dots,X,\\ z_2(t,x_0)=\beta_2(t),\quad t=t_1,t_1+1,\dots,t_2,\\ z_3(t_2,x)=z_2(t_2,x),\quad x=x_0,x_0+1,\dots,X,\quad z_3(t,x_0)=\beta_3(t),\quad t=t_2,t_2+1,\dots,t_3,\\ \alpha_1(x_0)=\beta_1(t_0),\ z_1(t_1,x_0)=\beta_2(t_1),\quad z_2(t_2,x_0)=\beta_3(t_2). \end{gathered}\quad (2)$$

Here, $D_i=\{(t,x): t=t_{i-1},t_{i-1}+1,\dots,t_i-1;\ x=x_0,x_0+1,\dots,X-1\}$, $i=\overline{1,3}$, where x_0, X, t_i, $i=\overline{1,3}$ are given, $f_i(t,x,a_i,b_i,u_i)$, $i=\overline{1,3}$ are the given n-dimensional vector-functions continuous in the aggregate of variables together with partial derivatives with respect to (a_i,b_i,u_i), $i=\overline{1,3}$ to the second order inclusively, $\alpha_1(x)$, $\beta_i(t)$, $i=\overline{1,3}$ are the given n-dimensional discrete vector-functions, and $u_i(t,x)$, $i=\overline{1,3}$ are r-dimensional vector-functions of control actions with values from the given non-empty, bounded and open sets $U_i\subset R^r$, $i=\overline{1,3}$, i.e.

$$u_i(t,x)\in U_i\subset R^r,\quad (t,x)\in D_i,\quad i=\overline{1,3}.\quad (3)$$

The triple $u(t,x)=(u_1(t,x),\,u_2(t,x),\,u_3(t,x))'$ with the properties mentioned above will be called an admissible control, its corresponding solution $z(t,x)=(z_1(t,x),\,z_2(t,x),\,z_3(t,x))'$ of boundary value problem (1)-(2) an admissible state of the process. Therewith, the pair $(u(t,x),\,z(t,x))$ is said to be an admissible process.

The problem is to minimize the functional

$$S(u)=\sum_{i=1}^{3}\varphi_i(z_i(t_i,X)),\quad (4)$$

determined on the solution of problem (1)-(2) generated by all-possible admissible controls.

Here $\varphi_i(z_i)$, $i=\overline{1,3}$ are the given continuously-differentiable scalar functions.

The admissible process $(u(t,x),\,z(t,x))$ being a solution of problem (1)-(4) is said to be an optimal control.

Considering $(u(t,x),\,z(t,x))$ as a fixed admissible process, we introduce the following denotation:

$$H_i(t,x,z_i,a_i,b_i,u_i,\psi_i)=\psi_i'\,f_i(t,x,z_i,a_i,b_i,u_i),$$

$$\frac{\partial f_i[t,x]}{\partial a_i} = \frac{\partial f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x))}{\partial a_i},$$

$$\frac{\partial H_i[t,x]}{\partial z_i} \equiv \frac{\partial H_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i},$$

where $\psi_i = \psi_i(t,x)$, $i=\overline{1,3}$ are n-dimensional vector-functions of conjugated variables being the solutions of the problem

$$\psi_i(t-1,x-1) = \frac{\partial H_i[t,x]}{\partial z_i} + \frac{\partial H_i[t-1,x]}{\partial a_i} + \frac{\partial H_i[t,x-1]}{\partial b_i}, \quad i=\overline{1,3},$$

$$\psi_1(t_1-1,X-1) = \psi_2(t_1-1,X-1) - \frac{\partial \varphi_1(z_1(t_1,X))}{\partial z_1},$$

$$\psi_1(t_1-1,x-1) = \psi_2(t_1-1,x-1) + \frac{\partial H_1[t_1-1,x]}{\partial a_1} - \frac{\partial H_2[t_1-1,x]}{\partial a_2}, \ \psi_1(t-1,X-1) =$$

$$= \frac{\partial H_1[t-1,X-1]}{\partial b_1}, \quad \psi_2(t_2-1,X-1) = \psi_3(t_2-1,X-1) - \frac{\partial \varphi_2(z_2(t_2,X))}{\partial z_2},$$

$$\psi_2(t_2-1,x-1) = \psi_3(t_2-1,x-1) + \frac{\partial H_2[t_2-1,x]}{\partial a_2} - \frac{\partial H_3[t_2-1,x]}{\partial a_3}, \tag{5}$$

$$\psi_2(t-1,X-1) = \frac{\partial H_2[t,X-1]}{\partial b_2}, \quad \psi_3(t_3-1,X-1) = -\frac{\partial \varphi_3(z_3(t_3,X))}{\partial z_3},$$

$$\psi_3(t_3-1,x-1) = \frac{\partial H_3[t_3-1,x]}{\partial a_3}, \quad \psi_3(t-1,X-1) = \frac{\partial H_3[t,X-1]}{\partial b_3}.$$

Theorem 1. For optimality of the admissible control $u(t,x)$, in problem (1)-(4) the relations

$$\frac{\partial H_i[\theta,\xi]}{\partial u_i} = 0, \quad (\theta,\xi) \in D_i, \ i=\overline{1,3}. \tag{6}$$

should be fulfilled.

Relation (6) as a necessary optimality condition of first order is the analogy of the Euler equation for the considered problem.

Using non-negativity condition of the second variation of the quality test, one can obtain second order necessary optimality conditions.

To this end, assume that in system (1)

$$f_i(t,x,z_i,a_i,b_i,u_i) = B_i(t,x)b_i + F(t,x,z_i,a_i,u_i). \tag{7}$$

Assume

$$K_1(\tau,s) = -R_1'(t_1,X;\theta,\tau)\frac{\partial^2 \varphi_1(z_1(t_1,X))}{\partial z_1^2}R_1(t_1,X;\theta,s) - Q_1'(t_2,X;\theta,\tau)\times$$

$$\times \frac{\partial^2 \varphi_2(z_2(t_2,X))}{\partial z_2^2}Q_2(t_2,X;\theta,s) - Q_3'(t_3,X;\theta,\tau)\frac{\partial^2 \varphi_3(z_3(t_3,X))}{\partial z_3^2}Q_3(t_3,X;\theta,s) +$$

$$+\sum_{t=\theta+1}^{t_1-1}\sum_{x=\max(\tau,s)+1}^{X-1}\left[R_1'(t,x;\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial z_1^2}R_1(t,x;\theta,s) + R_1'(t,x;\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial z_1 \partial a_1}\times\right.$$

$$\left.\times R_1(t+1,x;\theta,s) + R_1'(t+1,x;\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1 \partial z_1}R_1(t,x;\theta,s)\right]+$$

$$+\sum_{t=\theta}^{t_1-1}\sum_{x=\max(\tau,s)+1}^{X-1} R_1'(t+1,x;\theta,\tau)\frac{\partial^2 H_1[t,x]}{\partial a_1^2}R_1(t+1,x;\theta,s)+$$

$$+\sum_{t=t_1}^{t_2-1}\sum_{x=\max(\tau,s)+1}^{X-1}\Bigg[Q_1'(t,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}Q_1(t,x;\theta,s)+Q_1'(t,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}\times$$

$$\times Q_1(t+1,x;\theta,s)+Q_1'(t+1,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}Q_1(t,x;\theta,s)+Q_1'(t+1,x;\theta,\tau)\times$$

$$\times\frac{\partial^2 H_2[t,x]}{\partial a_2^2}Q_1(t+1,x;\theta,s)\Bigg]+\sum_{t=t_2}^{t_3-1}\sum_{x=\max(\tau,s)+1}^{X-1}\Bigg[Q_2'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_2(t,x;\theta,s)+$$

$$+Q_2'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}Q_2(t+1,x;\theta,s)+Q_2'(t+1,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}Q_2(t,x;\theta,s)+$$

$$+Q_2'(t+1,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_2(t+1,x;\theta,s)\Bigg],$$

$$K_2(\tau,s)=-R_2'(t_2,X;\theta,\tau)\frac{\partial^2\varphi_2(z_2(t_2,X))}{\partial z_2^2}R_2(t_2,X;\theta,s)-Q_3'(t_3,X;\theta,\tau)\times$$

$$\times\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}Q_3(t_3,X;\theta,s)+\sum_{t=\theta+1}^{t_2-1}\sum_{x=\max(\tau,s)+1}^{X-1}\Bigg[R_2'(t,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2^2}R_2(t,x;\theta,s)+$$

$$+R_2'(t,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial z_2\,\partial a_2}R_2(t+1,x;\theta,s)+R_2'(t+1,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2\,\partial z_2}R_2(t,x;\theta,s)\Bigg]+$$

$$+\sum_{t=\theta}^{t_2-1}\sum_{x=\max(\tau,s)+1}^{X-1}R_2'(t+1,x;\theta,\tau)\frac{\partial^2 H_2[t,x]}{\partial a_2^2}R_2(t+1,x;\theta,s)+$$

$$+\sum_{t=t_2}^{t_3-1}\sum_{x=\max(\tau,s)+1}^{X-1}\Bigg[Q_3'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}Q_3(t,x;\theta,s)+Q_3'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\times$$

$$\times Q_3(t+1,x;\theta,s)+Q_3'(t+1,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}Q_3(t,x;\theta,s)+Q_3'(t+1,x;\theta,\tau)\times$$

$$\times\frac{\partial^2 H_3[t,x]}{\partial a_3^2}Q_3(t+1,x;\theta,s)\Bigg],$$

$$K_3(\tau,s)=-R_3'(t_3,X;\theta,\tau)\frac{\partial^2\varphi_3(z_3(t_3,X))}{\partial z_3^2}R_3(t_3,X;\theta,s)+$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=\max(\tau,s)+1}^{X-1}\Bigg[R_3'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3^2}R_3(t,x;\theta,s)+R_3'(t,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial z_3\,\partial a_3}\times$$

$$\times R_3(t+1,x;\theta,s)+R_3'(t+1,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3\,\partial z_3}R_3(t,x;\theta,s)\Bigg]+$$

$$+\sum_{t=\theta}^{t_3-1}\sum_{x=\max(\tau,s)+1}^{X-1}R_3'(t+1,x;\theta,\tau)\frac{\partial^2 H_3[t,x]}{\partial a_3^2}R_3(t+1,x;\theta,s).$$

Here, $R_i(t,x;\tau,s)$, $i=\overline{1,3}$ are $(n\times n)$ matrix functions being the solutions of the following problems:

$$R_i(t,x;\,\tau-1,s-1)=R_i(t,x;\,\tau,s)\frac{\partial f_i[\tau,s]}{\partial z_i}+R_i(t,x;\,\tau-1,s)\frac{\partial f_i[\tau-1,s]}{\partial a_i}+$$
$$+R_i(t,x;\,\tau,s-1)\frac{\partial f_i[\tau,s-1]}{\partial b_i}, \tag{8}$$

$$R_i(t,x;\,t-1,s-1)=R(t,x;t-1,s)\frac{\partial f_i[t-1,s]}{\partial a_i},$$
$$R_i(t,x;\,\tau-1,x-1)=R(t,x;\tau,x-1)\frac{\partial f_i[\tau,x-1]}{\partial b_i}, \tag{9}$$
$$R_i(t,x;\,t-1,x-1)=E,$$

and $Q_i(t,x;\,\tau,s)$, $i=\overline{1,3}$ are determined by the formulas

$$Q_1(t,x;\,\tau,s)=R_2(t,x;\,t_1-1,x-1)R_1(t_1,x;\,\tau,s)+$$
$$+\sum_{\beta=s+1}^{x-1}\left[R_2(t,x;\,t_1-1,\beta-1)-R_2(t,x;\,t_1-1,\beta)\frac{f_2[t_1-1,\beta]}{\partial a_2}\right]R_1(t_1,\beta;\tau,s),$$
$$Q_2(t,x;\,\tau,s)=R_3(t,x;\,t_2-1,x-1)Q_1(t_2,x;\,\tau,s)+$$
$$+\sum_{\beta=s+1}^{x-1}\left[\left[R_3(t,x;\,t_2-1,\beta-1)-R_3(t,x;\,t_2-1,\beta)\right]\frac{f_3[t_2-1,\beta]}{\partial a_3}\right]Q_1(t_2,\beta;\tau,s).$$
$$Q_3(t,x;\,\tau,s)=R_3(t,x;\,t_2-1,x-1)R_2(t_2,x;\,\tau,s)+$$
$$+\sum_{\beta=s+1}^{x-1}\left[\left[R_3(t,x;\,t_2-1,\beta-1)-R_3(t,x;\,t_2-1,\beta)\right]\frac{f_3[t_2-1,\beta]}{\partial a_3}\right]R_2(t_2,\beta;\tau,s).$$

Using discrete variants of line variations [3, 4], we prove

Theorem 2. If the sets U_i, $i=\overline{1,3}$ are convex, then under the made assumptions for optimality of the classic extremal $u(t,x)$, in problem (1)-(4), (7) the following conditions

$$\sum_{\tau=x_0}^{X-1}\sum_{s=x_0}^{X-1}v_i'(\tau)\frac{\partial f_i'[\theta,\tau]}{\partial u_i}K_i(\tau,s)\frac{\partial f_i[\theta,s]}{\partial u_i}v_i(s)+$$
$$+2\sum_{x=x_0}^{X-1}\left[\sum_{s=x_0}^{x-1}v_i'(x)\frac{\partial^2 H_i[\theta,x]}{\partial u_i\,\partial a_i}R_i(\theta+1,x;\,\theta,s)\frac{\partial f_i[\theta,s]}{\partial u_i}v_i(s)\right]\le 0$$

should be fulfilled for all $v_i(x)\in U_i$, $x=x_0,x_0+1,\ldots,X-1$, $\theta\in T_i$, $i=\overline{1,3}$.

Reference

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