SECOND ORDER OPTIMALITY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS WITH NON-LOCAL CONDITIONS

Yagub Sharifov

Cybernetics Institute of ANAS, Baku, Azerbaijan
sharifov22@rambler.ru

Problem statement. The investigation object of the present paper is optimal control problems in systems of first order nonlinear ordinary differential equations with the boundary conditions:

$$\dot{x} = f(x,u,t), \quad x(t) \in R^n, \; t \in T = [t_0, t_1], \tag{1}$$

$$x(t_0) + Bx(t_1) = C. \tag{2}$$

Here $f(x,u,t)$ is the given n - dimensional vector-function continuous in totality of variables together with respect to x up to the second order inclusively; $A, \mathbf{B} \in R^{n\times n}$, $C \in R^{n\times 1}$ are constant matrices, $u(t)$ is r-dimensional measurable and bounded vector of controlling effects on the segment T .

It is assumed that almost everywhere on this segment the controlling effects satisfy the boundedness of the type of the inclusion:

$$u(t) \in U, \; t \in T, \tag{3}$$

where U is a open set from the space R^r .

The goal of the optimal control problem is the minimization of the functional

$$J(u) = \varphi(x(t_0), x(t_1)) + \int_T F(x,u,t)dt \tag{4}$$

determined in the solutions of boundary value problem (1), (2) for admissible control satisfying the condition (3). Here we assume that the scalar functions $\varphi(x,y)$ and $F(x,u,t)$ are continuous by their arguments and have continuous and bounded partial derivatives with respect to x and y up to the second order inclusively.

Let under some conditions the boundary value problem (1),(2) for each admissible process $u(t) \in U,\; t \in T$ have a unique solution $x(t,u)$. Admissible control $\{u(t), x(t,u)\}$, being a solution of the problem (1)-(4) i.e. delivering minimum to the functional (4) under restrictions (1)-(3) will be said to be optimal process, and $u(t)$-an optimal control.

2. Increment formula of the functional. We can carry out investigation of the optimal control problem (1)-(4) with using different variants of increment formula of the aim functional in two admissible processes $\{u, x\}$ and $\{\tilde{u} = u + \Delta u, \tilde{x} = x + \Delta x\}$. L.T. Rozonoer's [1] classic method of increments allows to obtain in this paper necessary optimality condition of the Pontryagin's maximum principle type [2]. For obtaining necessary optimality conditions of the second order for the Cauchy problem there are methods to obtain second order increment formulae suggested in [3,4]. In this section we'll obtain increment formulae for the second order functional for the problem (1)-(4) based on [4]. Notice that in deriving necessary optimality conditions, locality of the increment formula is essential, since the remainder terms are estimated by the quantity characterizing the smallness of the degree of the variation of control.

Necessary optimality conditions for an optimal control problem described by systems of ordinary differential equations have also been obtained in the paper [5-8].

Let $\{u, x = x(t,u)\}$ and $\{\tilde{u} = u + \Delta u,\; \tilde{x} = x + \Delta x = x(t,\tilde{u})\}$ be two admissible processes. We can define the boundary value problem in increments for the problem (1)-(2):

$$\Delta\dot{x} = \Delta f(x,u,t), \ t \in T$$
$$\Delta x(t_0) + B\Delta x(t_1) = 0$$

where by
$$\Delta f(x,u,t) = f(\tilde{x},\tilde{u},t) - f(x,u,t)$$
we denote a complete increment of the function $f(x,u,t)$.
We can represent the increment of the functional in the form:
$$\Delta J(u) = J(\tilde{u}) - J(u) = \Delta\varphi(x(t_0), x(t_1)) + \int_T \Delta F(x,u,t)dt$$

Let's introduce the Pontryagin's function
$$H(\psi, x,u,t) = \langle \psi(t), f(x,u,t) \rangle - F(x,u,t)$$
Require the vector function $\psi = \psi(t) \in R^n$ and constant vector $\lambda \in R^n$ be the solutions of the following conjugation problem (stationary state condition of Lagrange function by the state):
$$\dot{\psi}(t) = -\frac{\partial H(\psi, x,u,t)}{\partial x}, \quad t \in T$$
$$\psi(t_0) = \frac{\partial\varphi}{\partial x(t_0)} + \lambda,$$
$$\psi(t_1) = \frac{\partial\varphi}{\partial x(t_1)} + B'\lambda$$
Let the matrix function $\Phi(t), \ t \in T$ be a solution of the following matrix deferential equation
$$\dot{\Phi}(t) = \frac{\partial f(x,u,t)}{\partial x}\Phi(t)$$
with initial condition
$$\Phi(t_0) = E$$
where E is a unit matrix of dimension $n \times n$.
For the increment of the functional we get the terminal formula
$$\Delta J(u) = -\int_T \left\langle \frac{\partial H(\psi, x,u,t)}{\partial u}, \Delta u(t) \right\rangle dt - \frac{1}{2}\int_T \left\langle \Delta u(t)' \frac{\partial^2 H(\psi, x,u,t)}{\partial u^2}, \Delta u(t) \right\rangle dt$$
$$-\int_T \left\langle \Delta u(t)' \frac{\partial H^2(\psi, x,u,t)}{\partial x \partial u} + \frac{1}{2}\Delta x'(t)\frac{\partial^2 H(\psi, x,u,t)}{\partial x^2}, \Delta x(t) \right\rangle dt +$$
$$+\frac{1}{2}\left\langle \Delta x(t_0)' \frac{\partial^2\varphi}{\partial x(t_0)^2} + \Delta x(t_1)' \frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}, \Delta x(t_0) \right\rangle +$$
$$+\frac{1}{2}\left\langle \Delta x(t_0)' \frac{\partial^2\varphi}{\partial x(t_1)\partial x(t_0)} + \Delta x(t_1)' \frac{\partial^2\varphi}{\partial x(t_1)^2}, \Delta x(t_1) \right\rangle + \eta_{\tilde{u}}$$

3. Necessary optimality conditions. Let's $\Delta u(t) = \varepsilon\delta u(t)$. Then $\Delta x(t) = \varepsilon\delta x(t) + o(\varepsilon)$, and $\delta x(t)$-solution of variation equation corresponding (1)-(3). We can rewrite increment formula of functional in the form
$$\Delta J(u) = \varepsilon\delta J(u) + \frac{1}{2}\varepsilon^2\delta^2 J(u) + o(\varepsilon^2)$$

Theorem1. Let $\{u^*, x^*\}$ be an optimal process in the problem (1)-(4). Then this process satisfies almost everywhere on T the condition

$$H_u\left(\psi^*(\theta), x^*(\theta), u^*(\theta), \theta\right) \le 0 \tag{5}$$

for all $\theta \in [t_0, t_1]$ is an arbitrary tame point of the control $u(t)$.

Inequality (5) is the first order necessary optimality condition. This condition gives restricted information on controls that are suspicious for optimality. There are cases when condition (5) is fulfilled in a trivial way, i.e. it degenerates. In these cases it is desirable to have new optimality conditions allowing revealing non-optimality of those admissible controls for which the Pontryagin's maximum principle degenerates.

Definition. The admissible control $u(t)$ is said to be singular in the classical sense, if for all $\theta \in [t_0, t_1]$

$$H_u\left(\psi(\theta), x(\theta), u(\theta), \theta\right) = 0 \ . \tag{6}$$

Now let's get second order necessary optimality condition, when degenerates.
We introduce the function

$$
\begin{aligned}
R(\tau,s) = {} & \Phi^{-1'}(\tau)\left[\Phi(t_1)(E+B\Phi(t_1))^{-1}\right]' \frac{\partial^2\varphi}{\partial x(t_1)^2}\Phi(t_1)(E+B\Phi(t_1))^{-1}\Phi^{-1}(s) + \\
& + \Phi^{-1'}(\tau)\left[(E+B\Phi(t_1))^{-1}B\Phi(t_1)\right]' \frac{\partial^2\varphi}{\partial x(t_0)^2}(E+B\Phi(t_1))^{-1}B\Phi(t_1)\Phi^{-1}(s) + \\
& + \Phi^{-1'}(\tau)\left[(E+B\Phi(t_1))^{-1}B\Phi(t_1)\right]' \frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}\Phi(t_1)(E+B\Phi(t_1))^{-1}\Phi^{-1}(s) + \\
& + \Phi^{-1'}(\tau)\left[(E+B\Phi(t_1))^{-1}\Phi(t_1)\right]' \frac{\partial^2\varphi}{\partial x(t_0)\partial x(t_1)}\Phi(t_1)(E+B\Phi(t_1))^{-1}B\Phi^{-1}(s) + \\
& + \Phi^{-1'}(\tau)\left[(E+B\Phi(t_1))^{-1}B\Phi(t_1)\right]' \times \\
& \times \int_T \Phi'(t)\frac{\partial^2 H(\psi,x,u,t)}{\partial x^2}\Phi(t)dt(E+B\Phi(t_1))^{-1}B\Phi(t_1)\Phi^{-1}(s) + \\
& + \Phi^{-1'}(\tau)\int_{\max(\tau,s)}^{t_1}\Phi'(t)\frac{\partial^2 H(\psi,x,u,\tau)}{\partial x^2}\Phi(t)dt\Phi^{-1}(s) - \\
& - \Phi^{-1'}(\tau)\int_\tau^{t_1}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)d\xi(E+B\Phi(t_1))^{-1}B\Phi(t_1)\Phi^{-1}(s) - \\
& - \Phi^{-1}(\tau)\left[(E+B\Phi(t_1))^{-1}B\Phi(t_1)\right]'\int_\tau^{t_1}\Phi'(\xi)\frac{\partial^2 H(\psi,x,u,\xi)}{\partial x^2}\Phi(\xi)d\xi\Phi^{-1}(s)
\end{aligned}
$$

Theorem2. For the optimality of the singular control $u(t)$ in the problem (1)-(4) the inequality should be fulfilled

$$\delta^2 J(u) = -\left\{ \int_T \int_T \left\langle \delta' u(\tau) \frac{\partial' f(x,u,\tau)}{\partial u} R(\tau,s) \frac{\partial f(x,u,s)}{\partial u}, \delta u(s) \right\rangle d\tau ds + \right.$$

$$+ \int_T \left\langle \delta' u(\tau) \frac{\partial^2 H(\psi,x,u,t)}{\partial u^2}, \delta u(t) \right\rangle dt +$$

$$+ 2 \int_T \int_T \left\langle \delta u'(t) \frac{\partial^2 H(\psi,x,u,t)}{\partial x \partial u} \Phi(t) \left(E + B\Phi(t_1)\right)^{-1} \Phi^{-1}(s) \frac{\partial f(x,u,s)}{\partial u}, \delta u(s) \right\rangle dt ds +$$

$$\left. + \int_T \left\langle \int_t^{t_2} \delta u'(\tau) \frac{\partial^2 H(\psi,x,u,\tau)}{\partial x \partial u} \Phi(\tau) d\tau \Phi^{-1}(t) \frac{\partial f(x,u,t)}{\partial u}, \delta u(t) \right\rangle dt \right\} \geq 0$$

Theorem 3. For the optimality of the singular control $u(t)$ in the problem (1)-(4) inequality should fulfilled

$$\nu' \left\{ \int_T \int_T \left\langle \frac{\partial f(x,u,t)}{\partial u} R(t,s), \frac{\partial f(x,u,s)}{\partial u} \right\rangle dt ds + \right.$$

$$+ 2 \int_T \int_T \left\langle \frac{\partial^2 H(\psi,x,u,t)}{\partial x \partial u} \Phi(t) \left(E + B\Phi(t_1)\right) - 1 \Phi^{-1}(s), \frac{\partial f(x,u,s)}{\partial u} \right\rangle dt ds +$$

$$\left. + \int_T \left\langle \int_t^{t_2} \frac{\partial^2 H(\psi,x,u,t)}{\partial x \partial u} \Phi(\tau) d\tau \Phi^{-1}(t), \frac{\partial f(x,u,t)}{\partial u} \right\rangle dt \right\} \nu \leq 0$$

for all $\nu \in U$.

References

[1]. L.I.Rozonoer, *L.S. Pontryagin's maximum principle in the theory of optimal systems.* 1-3 //Avtomatika i telemekhanika. 1959, v. 20-22, No 10-12, 1320-1334 p., 1441-1456, 1561-1578. (Russian).

[2]. Gabasov R., Kirillova F.M. *The maximum principle in the theory of optimal control.* Minsk, Nauka tekhnika, 1974, 271 pp. (Russian).

[3]. Gabasov R., Kirillova F.M. *Singular optimal control*. Moscov, Nauka, 1973,256 pp. (Russian).

[4]. Mansimov K.B. *Singular controls in delay systems*. Baku ELM, 1999, 172 pp.(Russian).

[5]. Vasil'eva O.O., Misukami K. *Optimal contro of boundary value problem. //* Izvestia vysshikli uchebnikh zavedeniy. Matematika, 1994, No 12 (391), 33-41 p. (Russian).

[6]. Vasil'eva O.O., Misukami K. *Dynamical processes described by a boundary value problem: necessary optimality conditions and solution methods.* //Izv. RAN. Teoria and systemy upravlenia, 2000,No 1, p.95-100. (Russian).

[7]. Olga Vasilieva. *Optimal control in the class of smooth and bounded functions*. 15th Triennial World Congress, Barcelona, Spain. 2002, IFAC.

[8]. R.A.Sardarova , Sharifov Ya. A. *On necessary optimality conditions for multi-point conditions systems.* //Izvestia NANA. Teoria and sistemy upravlenia. 2004. No 2, 66-70 p. (Russian).

[9]. V.M.Alekseev, V.M.Tikhomirov, S.V.Fomin. *Optimal control*. Moscow, Nauka, 1985, 429 p. (Russian).