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INVESTMENT OPTIMIZATION PROBLEM WITH SAVAGE'S RISK CRITERIA UNDER UNCERTAINTIES

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While solving practical optimization problems, it is necessary to take into account various kinds of uncertainty due to lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors etc. Therefore widespread use of discrete optimization models in the last decades inspired many specialists to investigate various aspects of ill-posed problems theory and, in particular, the stability issues. The main difficulty while studying stability of discrete optimization problems is discrete models complexity, because even small changes of initial data make a model behave in an unpredictable manner. There are a lot of papers (see, e.g., [1-5]) devoted to analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations.

The present work continues our investigations of quantitative characteristics of stability of vector problems of integer and Boolean programming (see, e.g., [5-8]). Here the vector variant of the portfolio optimization problem with Savage's minimax risk criteria is considered [9], i.e. the problem of financial investments management, based on Markovitz's «portfolio theory» [10] (see also [11]). We give the achievable lower and upper bounds of the stability radius for the problem, i.e. for the limit of the perturbation of Savage criteria's parameters, that do not lead to the appearance of new Pareto-optimal portfolios.

For the formulation of the vector problem of choosing the optimal portfolio of assets with the minimax risk criteria we introduce the following notations:

$N_n = \{1, 2, ..., n\}$ – assets (stocks, companies bonds, real estate etc.),

N_m – economic strategies of an investor,

R – three dimensional risk matrix (missed opportunities matrix) of size $m \times n \times s$ with elements r_{ijk} from $\mathbf{R}$,

r_{ijk} –risk quantity ($k \in N_s$) of an investor choosing strategy $i \in N_m$ and asset $j \in N_n$ with criterion k,

$x = (x_1, x_2, ..., x_n)^T \in X \subset \{0,1\}^n$ – investor's portfolio of assets,

$$x_j = \begin{cases} 1, & \text{if the investor chooses an asset } j; \\ 0 & \text{otherwise.} \end{cases}$$

It is supposed, that each investor's portfolio x from a given portfolio set X assures an expected total profit p and does not overstep the limits of an available capital c, i. e. for each portfolio $x = (x_1, x_2, ..., x_n)^T \in X$ following conditions

$$\sum_{j \in N_n} p_j \, x_j \geq p,$$

$$\sum_{j \in N_n} c_j x_j \leq c$$

hold, where p_j is the expected profit of asset j, c_j is the cost of asset j.

Along with matrix $R=[r_{ijk}]$ we use its two dimensional sections $R_k \in \mathbf{R}^{m\times n}$, $k \in N_s$.

Let the following vector function

$$f(x,R)=(f_1(x,R_1),\ f_2(x,R_2),\dots,\ f_s(x,R_s))$$

be defined over the set X with «bottleneck» criteria, i.e. with Savage's minimax risk (extreme pessimism) criteria [9]:

$$f_k(x,R_k)=\max_{i\in N_m}\sum_{j\in N_n} r_{ijk}x_j \rightarrow \min_{x\in X}, \qquad k\in N_s.$$

Under the portfolio optimization problem $Z^s(R)$ we understand the problem of finding Pareto set $P^s(R)$, consisting of Pareto-optimal (efficient) portfolios

$$P^s(R)=\{x\in X:\ \ P^s(x,R)=\varnothing\},$$

where

$$P^s(x,R)=\{x'\in X:\ \ x \underset{R}{\succ} x'\},$$

and symbol $\underset{R}{\succ}$ is a binary relation defined over the set X as follows:

$$x \underset{R}{\succ} x' \quad \Leftrightarrow \quad f(x,R)\geq f(x',R) \ \text{ и } \ f(x,R)\neq f(x',R).$$

As usual (see, for example, [5,12]), stability radius of the problem $Z^s(R),\ s\geq 1$ to perturbations of a risk matrix is defined as follows:

$$\rho^s(R)=\begin{cases}\sup\Xi, & if \quad \Xi\neq\varnothing,\\ 0, & if \quad \Xi=\varnothing,\end{cases}$$

where

$$\Xi=\{\varepsilon>0: \forall R'\in\Omega(\varepsilon) \quad (P^s(R+R')\subseteq P^s(R))\},$$

$$\Omega(\varepsilon)=\{R'\in\mathbf{R}^{m\times n\times s}:\ \ \|R'\|<\varepsilon\},$$

$$\|R'\|=\max\{|\,r'_{ijk}\,|:\ \ (i,j,k)\in N_m\times N_n\times N_s\}, \quad R'=[r'_{ijk}].$$

It is obvious, that in the case $X=P^s(R)$ the stability radius $\rho^s(R)=\infty$. Therefore we consider the case, where $X\neq P^s(R)$.

We introduce the following notations:

$$\varphi=\min_{x\in X\setminus P^s(R)}\ \max_{x'\in P^s(x,R)}\ \min_{k\in N_s}\ \min_{i'\in N_m}\ \max_{i\in N_m}\frac{R_{ik}x-R_{i'k}x'}{\|x+x'\|^*},$$

$$\psi = \min_{x \in X \setminus P^s(R)} \max_{x' \in P^s(x,R)} \min_{k \in N_s} \min_{i' \in N_m} \max_{i \in N_m} \frac{R_{ik}x - R_{i'k}x'}{\|x - x'\|^*},$$

where $R_{ik} = (r_{i1k}, r_{i2k}, ..., r_{ink})$ is the i-th row of matrix $R_k \in \mathbf{R}^{m \times n}$,

$$\|z\|^* = \sum_{j \in N_n} |z_j|, \; z = (z_1, z_2, ..., z_n)^T.$$

Theorem. *Let $X \neq P^s(R)$. Then the following bounds are true:*

$$\varphi \le \rho^s(R) \le \psi.$$

for any $s \in \mathbf{N}$ and for the stability radius $\rho^s(R)$ of the problem $Z^s(R)$

The problem $Z^s(R)$, $s \ge 1$, is called stable, if $\rho^s(R) > 0$. In addition, let us introduce the traditional Smale set $Sm^s(R)$, i.e. the set of strongly efficient portfolios

$$Sl^s(R) = \{x \in X : \bar{\exists} x' \in X \quad \forall k \in N_s \quad (f_k(x, R_k) > f_k(x', R_k)\}.$$

Corollary. *If $X \neq P^s(R)$, then for any $s \in \mathbf{N}$ the next statements are equivalent:*

(i) $Z^s(R)$ *is stable,*

(ii) $P^s(R) = Sl^s(R)$,

(iii) $\varphi > 0$.

Upper bound ψ of stability radius $\rho^s(R)$, indicated in Theorem, is attainable, since for $m = 1$ our problem $Z^s(R)$ is transformed in a vector (s-criteria) Boolean programming problem with linear criteria:

$$R_k x \to \min_{x \in X}, \qquad k \in N_s, \tag{1}$$

and the upper bound turns into the form

$$\rho^s(R) \le \psi = \min_{x \in X \setminus P^s(R)} \max_{x' \in P^s(x,R)} \min_{k \in N_s} \frac{R_k(x - x')}{\|x - x'\|^*},$$

where R_k is k-th row of matrix $R \in \mathbf{R}^{s \times n}$. It is known [5,12], that right-hand side of this ratio is the expression of the stability radius of problem (1). Therefore, if $m = 1$, we have $\rho^s(R) = \psi$, that assures the attainability of this upper bound.

It is also easy to see, that lower bound φ is attainable. Indeed, let equality $\|x + x'\|^* = \|x - x'\|^*$ be true for any $x \in X \setminus P^s(R)$ and any $x' \in P^s(x, R)$, then $\rho^s(R) = \varphi = \psi$.

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