RECURRENT OPTIMALITY CONDITIONS OF SINGULAR CONTROLS IN DELAY CONTROL SYSTEMS

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Among the great number of optimal control problems, the problems of singular controls in delay control systems draw attention of researchers [1]. Notice that the transformations method in variation spaces is most convenient for studying optimality of controls singular in the classic sense [2]. It is easily generalized to multi-dimensional singular controls [3].

The present paper is devoted to the investigation of controls singular in the classic sence and analogous results are obtained for a delay controls system [4]

Consider the system

$$\dot{x}(t) = f(x(t),u(t),u(t-h),t), t \in T = [t_0,t_1], \tag{1}$$

$$x(t_0) = x_0, u(t) = \varphi(t), t \in [t_0 - h, t_0), \tag{2}$$

where $x = (x_1,...x_n)'$ is n-vector of phase coordinates, $u = (u_1,...u_r)'$ is r-vector of control actions, the prime is transposition sign, $h = const > 0$,

Let U be an open set of r-dimensional Euclidean space E^r. By $\widetilde{C}(T,E^r)$ denote a class of piecewise continuous and sufficiently piecewise – smooth vector functions $u(t): T \to E^r$.

Each piecewise – continuous and piecewise – smooth function $u = u(t),\quad t \in T$ (i.e. $u(t) \in \widetilde{C}(T,E^r)$) accepting the values from U:

$$u(t) \in U, t \in T \tag{3}$$

is said to be an admissible control.

Problem. Minimize the functional

$$J(u) = \Phi(x(t_1)) \tag{4}$$

on trajectories of problem (1), (2), generated by admissible controls (3).

We'll assume the vector functions $f(x,u,\upsilon,t),(x,u,\upsilon,t) \in E^n \text{x} U \text{x} U \text{x} T$ and $\varphi(t)$, $t \in [t_0 - h, t_0]$ sufficiently smooth and sufficiently piecewise smooth, respectively (exact assumptions for their analytic properties will directly follow from the representation form of final results), the function $\Phi(x), x \in E^n$ is assumed to be continuous together with its partial derivatives up to second order inclusively.

Under these conditions, it is easy to show by the successive approximations method that each admissible control generates a unique sufficiently piecewise –continuous, smooth solution of problem (1), (2), that will be assumed determined everywhere on T. The pair $(u(x), x(t))$ generates an admissible process.

The admissible control $u(t), t \in T$, being the solution of problem (1)-(4), is said to be an optimal control, the process $(u(x), x(t))$ an optimal process.

2. We'll carry out investigation on the base of the second variation of the minimized functional (4). Let $(u(x), x(t))$ be a fixed admissible process. By the scheme of [5, pp.51-57] it is easy to establish that the conditions

$$\delta^1 J(u;\delta u(t)) = -\int_{t_0}^{t_1} \left[H_u'(t)\delta u(t) + H_\upsilon(t)\delta u(t-h)\right]dt = 0, \tag{5}$$

$$\begin{aligned}\delta^2 J(u;\delta u(t)) &= \delta x'(t_1)\Phi_{xx}(x(t_1))\delta x(t_1) - \int_{t_0}^{t_1}[\delta x'(t)H_{xx}(t)\delta x(t) + \\ &+ \delta u'(t)H_{uu}(t)\delta u(t) + \delta u'(t-h)H_{\upsilon\upsilon}(t)\delta u(t-h) + 2\delta x'(t)H_{xu}(t)\delta u(t) + \\ &+ 2\delta x'(t)H_{x\upsilon}(t)\delta u(t-h) + 2\delta u'(t)H_{u\upsilon}(t)\delta u(t-h)]dt \ge 0,\end{aligned} \tag{6}$$

are fulfilled along the optimal control $u(t), t \in T$ for all $\delta u(t) \in \widetilde{C}(T, E^r)$.

Here, $\delta^1 J(\cdot)$ is the first, $\delta^2 J(\cdot)$ is the second variation of the functional (4); $\delta u(t), t \in T$ is a variation of the control $u(t), t \in T$, $\delta x(t), t \in T$ is a variation of the trajectory $x(t), t \in T$ being a solution of the system

$$\begin{aligned}&\delta\dot{x}(t) = f_x(t)\delta x(t) + f_u(t)\delta u(t) + f_\upsilon(t)\delta u(t-h), t \in T, \\ &\delta x(t_0) = 0, \quad \delta u(t) = 0, \quad t \in [t_0 - h, t_0);\end{aligned} \tag{7}$$

$$H(\psi, x, u, \upsilon, t) = \psi' f(x, u, \upsilon, t),\ H(t) = H_\mu(\psi(t), x(t), u(t), \upsilon(t), t),$$

$\upsilon(t) = u(t-h),\ f_\mu(t) = f_\mu(x(t), u(t), \upsilon(t-h), t),\ t \in T$, $H(t) = 0,$ при $t > t_1$,

$$H_{\mu\nu}(t) = H_{\mu\nu}(\psi(t), x(t), u(t), \upsilon(t), t),\ \mu, \nu \in \{x, u, \upsilon\};$$

where $\psi(t)$ is a solution of the conjugated system

$$\dot{\psi}(t) = -H_x(t), t \in T,\ \psi(t_1) = -\Phi_x(x(t_1)), \psi(t) = 0, t > t_1. \tag{8}$$

(5), (6) yield the classical necessary optimality conditions (analogies of Euler equations and Legendre-Klebsh conditions) [1].

$$H_u(t) + H_\upsilon(t+h) = 0, \quad t \in T, \tag{9}$$

$$w'[H_{uu}(t) + H_{\upsilon\upsilon}(t+h)]w \le 0, \quad t \in T, \quad \forall w \in E^r. \tag{10}$$

Definition 1. [5]. The admissible control $u(t), t \in T$, satisfying (9) is said to be singular (in the classic sense) if

$$rang[H_{\upsilon\upsilon}(t) + H_{\upsilon u}(t+h)] = 0, \quad t \in T.$$

Our goal is to investigate such singular controls.

Introduce the matrix functions determined by the recurrent formulae:

$$\begin{aligned}&g_k[\mu](t) = f_x(t)g_{k-1}[\mu](t) - \frac{d}{dt}g_{k-1}[\mu](t), \quad k = 1,2,..., \\ &g_0[\mu](t) = f_\mu(t), \quad \mu \in \{u, \upsilon\};\end{aligned} \tag{11}$$

where

$$\begin{aligned}&\delta_k\dot{x}(t) = f_x(t)\delta x(t) + g_k[u](t)\delta_k u(t) + g_k[\upsilon](t)\delta_k u(t-h), \quad t \in T, \\ &\delta_k x(t_0) = 0, \quad \delta_k u(t) = 0, \quad t \in [t_0 - h,\ t_0), k = 0,1,2,...,\end{aligned} \tag{12}$$

$$\begin{aligned}&\delta_0 u(t) = \delta u(t), t \in T, \quad \delta_0 x(t) = \delta x(t), t \in T, \\ &\delta_k u(t) = \int_{t_0}^{t}\delta_{k-1}u(\tau)d\tau, \quad t \in T, \delta_k u(t) = 0, \quad t \in [t_0 - h, t_0], \quad k = 0,1,2,...,\end{aligned} \tag{13}$$

$$G_k[\mu](t) = H_{xx}(t)g_{k-1}[\mu](t) - f_x'(t)G_{k-1}[\mu](t) - \frac{d}{dt}G_{k-1}[\mu](t), \quad k = 1,2,..., \tag{14}$$

$$G_0[\mu](t) = H_{x\mu}(t), t \in T, \quad \mu \in \{u, \upsilon\};$$

$$L_k[\mu](t) = -g_{k-1}[\mu](t)H_{xx}(t)g_{k-1}[\mu](t) + 2g'_k[\mu](t)G_{k-1}[\mu](t) +$$
$$+\frac{d}{dt}[g'_{k-1}[\mu](t)G_{k-1}[\mu](t)], \quad k = 1,2,..., \tag{15}$$
$$L_0[\mu](t) = H_{\mu\mu}(t), \quad t \in T.$$
$$\Psi(s,\tau) = \int_s^{t_1} \lambda(s,t)H_{xx}(t)\lambda(\tau,t)dt - \lambda(s,t_1)\Phi_{xx}(x(t_1))\lambda(\tau,t_1), \quad (s,\tau) \in T \times T,$$

where $\frac{\partial \lambda(s,t)}{\partial t} = f_x(t)\lambda(s,t), \quad t_0 \le s < t < t_1, \; \lambda(s,s) = E, \quad \lambda(s,t) = 0, s > t$ (E – is a unique $n \times n$-matrix);

$R_k[u,\upsilon](s,\tau) = g'_k[u](s)\{\lambda'(s,\tau)G_k[\upsilon](\tau) + \Psi(s,\tau)g_k[\upsilon](\tau)\}, (s,\tau) \in T \times T, \quad k = 0,1,...,$

$Q_k[u,\upsilon](t) = g'_k[u](t)G_k[u](t) + g'_k[\upsilon](t+h)G_k[\upsilon](t+h), t \in T; \quad k = 0,1,....$

The following theorem is easily proved by means of the modified variant of transformation of variations [4,6] as a convenient method it uses a Legendre polynomial [7], and also takes into account (11)-(15) by the scheme from [8, pp.107-135]

Theorem. Let the conditions

$$L_m[u](t) + L_m[\upsilon](t+h) = 0, \forall t \in T, m = 0,1,...,k, \quad k \in \{0,1,2,...\}.$$

be fulfilled for the control $u(t), t \in T$ singular in the sense of definition 1.

Then for optimality of the control $u(t), t \in T$ the relations

$$\tilde{u}'\{R_m[u,u](\theta,\theta) + 2R_m[u,\upsilon](\theta,\theta+h) + R_m[\upsilon,\upsilon](\theta+h,\theta+h)\}\tilde{u} \le 0, m = 0,1,...,k;$$
$$Q_m[u,\upsilon](\theta) - Q'_m[u,\upsilon](\theta) = 0, m = 0,1,...,k;$$
$$\tilde{u}'\{L_{k+1}[u](\theta) + L_{k+1}[\upsilon](\theta+h)\}\tilde{u} \ge 0,$$

should be fulfilled for all $\theta \in [t_0, t_1), \tilde{u} \in E^r$.

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