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ON THE CONSTRUCTION OF A HOMOGENEOUS STANDARD MARKOV PROCESS

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1. Consider the inhomogeneous (with infinite lifetime) Markov process
$$X = \left(\Omega, M^s, M_t^s, X_t, P_{s,x}\right), \quad 0 \le s \le t < \infty,$$
in the state space (E, B), i.e. it is assumed that:

(1) E is a locally compact Hausdorff space with a countable base, □ is the σ-algebra for Borel sets of the space;

(2) for every $s \ge 0$, $x \in S$, $P_{s,x}$ is a probability measure on the σ-algebra M^s, M_t^s, $t \ge s$, is the increasing family of sub-σ-algebras of the σ-algebra M^s, where
$$M^{s_1} \supseteq M^{s_2}, \quad M_t^s \subseteq M_v^u \text{ for } s_1 \le s_2,\ u \le s \le t \le v,$$
it is assumed as well that
$$\overline{M}^s = M^s, \quad \overline{M}_t^s = M_t^s = M_{t+}^s,\ 0 \le s \le t < \infty,$$
where $\overline{M}^s$ is a completion of M^s with respect to the family of measures $\{P_{u,x},\ u \le s,\ x \in E\}$, $\overline{M}_t^s$ is the completion of M_t^s in $\overline{M}^s$ with respect to the same family of measures;

(3) the paths of the process $X = (X_t(\omega)),\ t \ge 0$, are right continuous on the time interval $[0, \infty)$;

(4) for each $t \ge 0$ the random variables $X_t(\omega)$ (with values in (E, B)) are M_t^s-measurable, $t \ge s$, where it is supposed that
$$P_{s,x}(\omega:\ X_s(\omega) = x) = 1$$
and the function $P_{s,x}(X_{s+h} \in B)$ is measurable in (s, x) for the fixed $h \ge 0$, $B \in \mathrm{B}$ (with respect to $\mathrm{B}[0, \infty) \otimes \mathrm{B}$);

(5) the process X is strong Markov: for every $\left(M_t^s,\ t \ge s\right)$-stopping time τ (i.e. $\{\omega:\ \tau(\omega) \le t\} \in M_t^s, t \ge s$) we should have
$$P_{s,x}\left(X_{\tau+h} \in B \middle| M_\tau^s\right) = P(\tau, X_\tau, \tau + h, B) \quad \left(\{\tau < \infty\},\ P_{s,x} - \text{a.s.}\right),$$
where
$$P(s, x, s + h, B) \equiv P_{s,x}(X_{s+h} \in B);$$

(6) the process X is quasi-left-continuous: for every non-decreasing sequence of $\left(M_t^s,\ t \ge s\right)$-stopping times $\tau_n \uparrow \tau$ should be
$$X_{\tau_n} \to X_\tau \quad \left(\{\tau < \infty\},\ P_{s,x} - \text{a.s.}\right)$$

Let $g(t, x)$ be the Borel measurable functions (i.e. measurable with respect to the product σ-algebra $\mathrm{B}' = \mathrm{B}[0, \infty) \otimes \mathrm{B}$) which is defined on $[0, \infty) \times E$ and takes its values in $(-\infty, +\infty]$.

Assume now the following integrability condition of a random process $g(t, X_t(\omega))$, $t \ge 0$:
$$E_{s,x} \sup_{t \ge s} g^-(t, X_t) < \infty,\ s \ge 0,\ x \in S. \tag{1}$$

The problem of optimal stopping for the process X with the gain $g(t,x)$ is stated as follows: the value-function (payoff) $v_T(s,x)$ is introduced as

$$v_T(s,x) = \sup_{\tau \in \mathrm{M}_s^T} E_{s,x} g(\tau, X_\tau), \tag{2}$$

where M_s^T is the class of all finite ($P_{s,x}$-a.s.) $\left(M_t^s,\ t \ge s\right)$-stopping times, $\tau \le T$, and it is required to find the stopping time τ_ε (for each $\varepsilon > 0$) for which

$$E_{s,x} g(\tau_\varepsilon, X_{\tau_\varepsilon}) \ge v_T(s,x) - \varepsilon$$

for any $x \in E$.

Such a stopping time is called ε-optimal, and in the case $\varepsilon = 0$ it is called simply an optimal stopping time.

To construct ε-optimal (optimal) stopping times it is necessary to characterize the value $v(s,x)$ and for this purpose the following notion of an excessive function turns out to be fundamental.

2. Let us introduce now the new space of elementary events $\Omega' = [0,\infty) \times \Omega$ with elements $\omega' = (s,\omega)$, the new state space (extended state space) $E' = [0,\infty) \times E$ with the σ-algebra $\mathrm{B}' = \mathrm{B}[0,\infty) \otimes \mathrm{B}$, the new random process X' with values in (E', B')

$$X'_t(\omega') = X'_t(s,\omega) = (s+t, X_{s+t}(\omega)), \quad s \ge 0, \quad t \ge 0,$$

and the translation operators Θ'_t:

$$\Theta'_t(s,\omega) = (s+t, \omega), \quad s \ge 0, \quad t \ge 0,$$

where it is obvious that

$$X'_u(\Theta'_t(\omega')) = \Theta'_{u+t}(\omega'), \quad u \ge 0, \quad t \ge 0.$$

Introduce in the space Ω' the σ-algebra:

$$N^0 = \sigma(X'_u,\ u \ge 0), \qquad N_t^0 = \sigma(X'_u,\ 0 \le u \le t)$$

and on the σ-algebra N^0 the probability measures

$$P'_{x'}(A) = P'_{(s,x)}(A) \equiv P_{s,x}(A_s),$$

where $A \in N^0$ and A_s is the section of A at the point s

$$A_s = \{\omega :\ (s,x) \in A\},$$

where it is easy to see that $A_s \in \Phi^s \equiv \sigma(X_u,\ u = s)$ and if $a \in N_t^0$, then

$$A_s \in F_{s+t}^s \equiv \sigma(X_u,\ s \le u \le s+t).$$

Consider the function

$$P'(h, x', B') \equiv P'_{x'}(X'_h \in B').$$

We have to verify that this function is measurable in x' for a fixed $h \ge 0$. For the rectangles $B' = \Gamma \times B$ which generate the σ-algebra B' we have

$$P'(h, x', B') \equiv P'_{s,x}(\omega :\ (s+h, X_{s+h}(\omega)) \in \Gamma \times B) = I_{(s+h \in \Gamma)} P_{s,x}(X_{s+h} \in B);$$

therefore for the rectangles the function $P'(h, x', B')$ is measurable in x'. Consider now the class of all sets B', $B' \in \mathrm{B}'$, for which the function $P'_{x'}(X'_h \in B')$ is B'-measurable in x'. It is easy to verify that this class of sets satisfies all the reqiurements of the monotone class theorem; therefore it coincides with the σ-algebra B'.

Thus the function $P'(h, x', B')$ is measurable in x', and hence we can introduce the measures $P'_{\mu'}$ on the σ-algebra N^0 for every finite measure μ' on (S', B') by averaging

$P'_{x'}$ with respect to μ' [3]. Let us perform the completion of σ-algebra N^0 with respect to the family of all measures $P'_{\mu'}$, denote this completion by N' and then perform the completion of each σ-algebra N_t^0 in N' with respect to the same family of measures denoting them by N'_t.

Lemma. *If $\tau'(\omega')$ is an N_{t+}^0-stopping time, then $\tau(\omega) = s + \tau'(s,\omega)$ is a $\left(\Phi_{t+}^s,\ t \ge s\right)$-stopping time, where $\Phi_t^s = \sigma(X_u,\ s \le u \le t),\ t \ge s$.*

Proof. Indeed, we have

$$(\omega:\ \tau(\omega) < t) = (\omega:\ \tau'(s,\omega) < t - s) = (\omega':\ \tau'(\omega') < t - s)_s,$$

but $(\omega':\ \tau'(\omega') < t - s) \in N_{t-s}^0$; therefore the section $(\omega':\ \tau'(\omega') < t - s)_s$ belongs to Φ_t^s. Thus $\tau(\omega)$ is a $\left(\Phi_{t+}^s,\ t \ge s\right)$- stopping time.

The following key result (in a somewhat different form) was proved in the paper [2].

Theorem. *The random process*

$$X = (\Omega', N', N'_t, X'_t, \Theta'_t, P'_{x'}),\quad t \ge 0,$$

is a homogeneous standard Markov process in the space (S', B').

Proof. The main step in the proof is to verify that the process $\left(\Omega', N^0, N_{t+}^0, X'_t, \Theta'_t, P'_{x'}\right)$, $t \ge 0$, is strong Markov, i.e. we have to show that

$$E'_{x'}\left[f'(X'_{\tau'+h}) \cdot I_{(\tau'<\infty)}\right] = E'_{x'}\left[M'_{X'_{\tau'}} f'(X'_h) \cdot I_{(\tau'<\infty)}\right], \tag{3}$$

where $f'(x')$ is an arbitrary bounded B'-measurable function and τ' is an arbitrary N_{t+}^0-stopping time. Using again the monotone class theorem, it is clear that this relation suffices to be proved for the indicator functions

$$f'(x') = I_{(s\in\Gamma)} \cdot I_{(x\in B)}.$$

Thus it is needed to check that

$$E_{s,x}\left[I_{(s+\tau'(s,\omega)+h\in\Gamma)} \cdot I_{(s+\tau'(s,\omega)+h\in B)} \cdots I_{(\tau'(s,\omega)<\infty)}\right] =$$

$$= E_{s,x}\left[I_{(s+\tau'(s,\omega)+h\in\Gamma)} P_{u,y}\left(X_{u+h} \in B\right)\Big|_{\substack{u=s+\tau'(s,\omega) \\ y=X_{s+\tau'}(s,\omega)}} \cdot I_{(\tau'(s,x)<\infty)} \right].$$

We know from Proposition 7.3, Ch. I in [3] that the strong Markov property (3) of the process X' remains true for arbitrary N'_t, $t \ge 0$-stopping times τ' and from Proposition 8.12, Ch. I in [3] we get that $N'_t = N'_{t+}$. The quasi-left-continuity of the process X' now easily follows from the same property of *X* with the help of Lemma. Theorem is proved.

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