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SM|M|∞ IN SPECIAL LIMIT CONDITIONS*

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In this paper the queueing system with input semi-markovian [1-4] process and unlimited number of servers is considered. The arrived demand occupies any of free servers. Durations of various demands services are stochastic independent, equally distributed to the exponential law with rate μ. After service, the demand leaves the system.

For a definition of semi-markovian process, we are considering two-dimensional homogeneous markovian process {ξ(*n*), τ(*n*)} with discrete time and semi-markovian matrix $A(x)$ with elements $A_{kv}(x)$, which defined by following equation.

$$A_{\nu k}(x) = P\left\{\xi(n+1) = k, \tau(n+1) < x \middle| \xi(n) = \nu\right\}. \quad (1)$$

Here ξ(*n*) is ercodic Markov chain with discrete time and matrix $P = [p_{vk}]$ probability of transition for one step [2], which defined by equation

$$P = A(\infty). \quad (2)$$

This chain for semi-markovian process is called embedded Markov chain. Process *τ*(*n*) accepts non-negative values from continuous set.

Stochastic process of homogeneous events

$$t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots$$

is called semi-markovian process (SM-process), set by matrix *A* (*x*) if for the moments t_n approaches of its events of the following equations are right

$$t_{n+1} = t_n + \tau(n+1).$$

In general case for the elements of semi-markovian matrix has the multiplicative form which can be written as

$$A_{\nu k}(x) = G_{\nu k}(x) p_{\nu k},$$

where $G_{vk}(x)$ is the conditional distribution function of an interval length of semi-markovian process on conditions that in the beginning of this interval the embedded Markov chain ξ(*n*) has value *v*, and at the end of it will accept value *k*. Matrix *A*(*x*) can be written as Hadamard product

$$A(x) = G(x) * P \quad (3)$$

two matrixes *G* (*x*) and *P*.

The main purpose of this paper is the finding of probability distribution of the server number in queueing system.

Let the system operates in a steady-state conditions [5-7]. Denote *i*(*t*) – the server number at the moment *t*, then stationary probability distribution of values of process *i*(*t*) denotes π (*i*) = *P* {*i* (*t*) = *i*}.

We consider three-dimensional process {*s* (*t*), *z* (*t*), *i* (*t*)} which is markovian [4]. Here *z*(*t*) – ia the length of an interval from time moment *t* till the moment of approach of the next event in the considered SM-process, and stochastic process *s*(*t*) with piecewise constant realizations continuous at the left, defined by equality

$$s(t) = \xi(n+1), \text{ if } t_n < t \le t_{n+1}.$$

From probability distribution

$$P(s,z,i,t) = P\left\{s(t) = s, z(t) < z, i(t) = i\right\}$$

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of markovian process $\{s\ (t), z\ (t), i(t)\}$ we pass on to one-dimensional marginal distribution

$$\pi(i) = \sum_s P(s,\infty,i) .$$

For marginal distributions it is easy to obtain the system of Kolmogorov differential equations which we rewrite for stationary probability distribution $P\ (s, z, i) = P\ (s, z, i, t)$ as

$$\frac{\partial P(s,z,i)}{\partial z} - \frac{\partial P(s,0,i)}{\partial z} - P(s,z,i)i\mu + P(s,z,i+1)\left(i+1\right)\mu + \sum_{\nu} \frac{\partial P(\nu,0,i-1)}{\partial z} A_{\nu s}(z) = 0. \quad (4)$$

Denote

$$H(s,z,u) = \sum_{i=0}^{\infty} e^{jui} P(s,z,i) , \quad (5)$$

where $j = \sqrt{-1}$ – imaginary unit.

For these functions we obtain differentially-matrix equation

$$\frac{\partial H(z,u)}{\partial z} + j\mu\left(1-e^{-ju}\right)\frac{\partial H(z,u)}{\partial u} + \frac{\partial H(0,u)}{\partial z}\left\{e^{ju}A(z) - I\right\} = 0 , \quad (6)$$

where

$$H(z,u) = \left\{H(1,z,u), H(2,z,u), \cdots\right\} .$$

Solution $H(z, u)$ of differentially-matrix equation (6) satisfies to the condition

$$H(z,0) = R(z) , \quad (7)$$

and define characteristic function of the $i\ (t)$ by equation

$$h(u) = Me^{jui(t)} = H(\infty,u)E .$$

where function $R(z)$ is the stationary probability distribution of two-dimensional markovian process $\{s(t), z(t)\}$.

The condition of limit rare changes states of SM-process

The queue SM | M | ∞ will investigated in asymptotic condition of limit rare changes states of input SM-process [6, 8, 9].

The condition of limit rare changes of states semi-markovian process is formalized by following equality for matrix P transitions probabilities of its embedded Markov chain

$$P = I + \delta \cdot Q , \quad (8)$$

where δ – some small rate ($\delta \to 0$), and I – an identity diagonal matrix.

Matrix Q with elements q_{vk} is similar to matrix infinitesimal characteristics and has the same properties, that is by $k \neq v$ matrix elements $q_{vk} > 0$, and also equals are right

$$\sum_k q_{\nu k} = 0 , \ \sum_{k\neq\nu} q_{\nu k} = -q_{kk} .$$

Semi-markovian matrix for SM-process in condition of limit rare changes states

$$A\left(x,\delta\right) = G\left(x\right) * \left\{I + \delta \cdot Q\right\} . \quad (9)$$

Solution of the differential equation (6) and vector of stationary probability distribution of process $\{s(t), z(t)\}$ in asymptotic condition of limit rare changes states denote accordingly $H(z, u, \delta)$ and $R(z, \delta)$.

Then according to the entered notation the differentially-matrix equation (6) and a condition (7) we will write down as

$$\begin{cases} \dfrac{\partial H(z,u,\delta)}{\partial z} + j\mu\left(1-e^{-ju}\right)\dfrac{\partial H(z,u,\delta)}{\partial u} + \dfrac{\partial H(0,u,\delta)}{\partial z}\left\{e^{ju}A(z,\delta) - I\right\} = 0, \\ H(z,0,\delta) = R(z,\delta). \end{cases} \quad (10)$$

Here vector $R(z, \delta)$ with element $R(s, z, \delta)$ is given by

$$R\left(z,\delta\right) = \kappa_1\left(\delta\right) r \int_0^z \left(P\left(\delta\right) - A(x,\delta)\right) dx , \quad (11)$$

where r is stationary probability distribution of embedded Markov chain, and value $\kappa_1(\delta)$ is defined by equation

$$\kappa_1(\delta) = \frac{1}{rA(\delta)E}, \tag{12}$$

and matrix $A(\delta)$ is given by

$$A(\delta) = \int_0^\infty \left(P(\delta) - A(x,\delta)\right)dx. \tag{13}$$

According to Poincare's theorem [10] about analytical dependence of the solution on the rate there is a limit

$$\lim_{\delta\to 0} H(z,u,\delta) = F(z,u), \tag{14}$$

Then in the system (10), according to (14), we realize limiting transition at $\delta \to 0$. For functions $F(s, z, u)$, according to a view of matrix $A(z)$, we obtain the set of independent differential equations

$$\begin{cases} \dfrac{\partial F(s,z,u)}{\partial z} + j\mu\left(1-e^{-ju}\right)\dfrac{\partial F(s,z,u)}{\partial u} + \dfrac{\partial F(s,0,u)}{\partial z}\left\{e^{ju}G_{ss}(z)-1\right\} = 0, \\ F(s,0,u) = R(s,z), \end{cases} \tag{16}$$

where

$$R(z) = \lim_{\delta\to 0} R(z,\delta).$$

Condition of a growing service time
Asymptotic of the first order

The equation (16) will be solved in asymptotic condition of a growing service time [4, 6, 7], let $\mu \to 0$. Denote $\mu = \varepsilon$ and in differential equation of the system (16) replacements are realized

$$u = \varepsilon w, \quad F(s,z,u) = F_1(s,z,w,\varepsilon), \tag{17}$$

for $F_1(s, z, w, \varepsilon)$ obtain equation

$$\frac{\partial F_1(s,z,w,\varepsilon)}{\partial z} + j\left(1-e^{-j\varepsilon w}\right)\frac{\partial F_1(s,z,w,\varepsilon)}{\partial w} + \frac{\partial F_1(s,0,w,\varepsilon)}{\partial z}\left\{e^{j\varepsilon w}G_{ss}(z)-1\right\} = 0. \tag{18}$$

We consider such class of solution $F(s, z, u)$ the equations (16) for which, in view of (22), functions $F_1(s, z, w, \varepsilon)$ possess following properties: there are finite limits at $\varepsilon \to 0$ of functions $F_1(s, z, w, \varepsilon)$ and their derivatives on w, on z and on z in zero.

Theorem1. Solution of the differential equation (18) at $\varepsilon \to 0$ is given by

$$F_1(s,z,w) = R(s,z)\exp\{jw\kappa_1\}, \tag{19}$$

where stationary probability distribution $R(z)$ of two-dimensional markovian process $\{s(t), z(t)\}$ is given by

$$R(z) = \lim_{\delta\to 0} R(z,\delta) = \kappa_1 r\int_0^z \left(I - A(x)\right)dx.$$

Here $A(x) = \lim_{\delta\to 0} A(x,\delta)$ – semi-markovian matrix, r – stationary probability distribution of embedded Markov chain, and value κ_1 is defined by equality

$$\kappa_1 = \lim_{\delta\to 0}\kappa_1(\delta) = \frac{1}{rAE},$$

where matrix A is defined as

$$A = \int_0^\infty \left(I - A(x)\right)dx.$$

Corollary. Function

$$h_1(u) = \exp\{ju\kappa_1/\mu\}$$

we will be called an asymptotic of the first order of characteristic function $h(u)$ of the process $i(t)$.

The asymptotic of second order

For more detailed investigation of considered queue system, we will obtain the asymptotic of the second order.

In the differential equation of the system (16) we realize the replacement

$$F(s,z,u) = F^{(2)}(s,z,u)\exp\{ju\kappa_1/\mu\}. \quad (20)$$

There have

$$\frac{\partial F^{(2)}(s,z,u)}{\partial z} + j\mu\left(1-e^{-ju}\right)\left\{\frac{\partial F^{(2)}(s,z,u)}{\partial u} + \frac{j\kappa_1}{\mu}\right\} + \frac{\partial F^{(2)}(s,0,u)}{\partial z}\left\{e^{ju}G_{ss}(z)-1\right\} = 0. \quad (21)$$

Denote $\varepsilon^2 = \mu$, in equation (21) realize replacement

$$u = \varepsilon w, \quad F^{(2)}(s,z,u) = F_2(s,z,w,\varepsilon), \quad (22)$$

and obtain

$$\frac{\partial F_2(s,z,w,\varepsilon)}{\partial z} + j\varepsilon\left(1-e^{-jw\varepsilon}\right)\frac{\partial F_2(s,z,w,\varepsilon)}{\partial w} - \kappa_1\left(1-e^{-jw\varepsilon}\right) + \frac{\partial F_2(s,0,w,\varepsilon)}{\partial z}\left\{e^{jw\varepsilon}G_{ss}(z)-1\right\} = 0. \quad (23)$$

We consider such class of solution $F^{(2)}(s, z, u)$ the equations (21) for which, in view of (22), functions $F_2(s, z, w, \varepsilon)$ possess following properties: there are finite limits at $\varepsilon \to 0$ of functions $F_2(s, z, w, \varepsilon)$ and their derivatives on w, on z and on z in zero.

Theorem 2 Solution of the differential equation (23) at $\varepsilon \to 0$ is given by

$$F_2(s,z,w) = R(s,z)\exp\left\{\frac{(jw)^2}{2}\kappa_2\right\}, \quad (24)$$

where value κ_2 is defined by equation

$$\kappa_2 = \frac{\partial f_2(0)}{\partial z}E, \quad (25)$$

and vector-function $f_2(z)$ satisfies to condition $f_2(\infty)\,E = 0$ and is the equation solution

$$\frac{\partial f_2(s,z)}{\partial z} + \frac{\partial f_2(s,0)}{\partial z}\left(G_{ss}(z)-1\right) + \frac{\partial R(s,0)}{\partial z}G_{ss}(z) - \kappa_1 R(s,z) = 0. \quad (26)$$

Corollary. Function

$$h_2(u) = \exp\left\{\frac{ju\kappa_1}{\mu} + \frac{(ju)^2}{2}\frac{\kappa_2}{\mu}\right\}$$

we will be called the asymptotic of the second order of characteristic function $h(u)$ of the 0 process $i(t)$.

Conclusion

In this paper is found the asymptotic probability distribution of the server number in queue SMlMl∞ in the condition of limit rare changes of conditions in the SM-process and the condition of growing service time. The obtained distribution can be multimodal.

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