THE RESEARCH OF RQ-SYSTEM WITH INPUT MMP PROCESS*

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We consider single-line RQ-system (Retrial queue) with a recall source and incoming markovian modulate Poisson process (MMP-process) set by matrix of infinitesimal characteristics Q and intensity λ_n..

Suppose, that the demand which has found the service free, occupies it for service during the random time distributed to the exponential law with rate μ. If the device is occupied, the arrived demand passed in a recall source in which carries out the random delay which duration has exponential distribution with rate σ. After that demand become to service with retrial of its capture. If the service is free, the demand from the recall source occupies it on a random in-service time, if it is occupied the demand instantly comes back in the recall source for realization of the following delay of random duration.

Let $i(t)$ – demands number in the recall source, $n(t)$ – state of Markov chain in MMP-process, and $k(t)$ – state of service set, i.e.:

$$k(t)=\begin{cases}0, \text{ service is free,}\\ 1, \text{ service is buzy.}\end{cases}$$

Denote

$$P\{k(t)=k,\, n(t)=n,\, i(t)=i\}=P\{k,n,i,t\}$$

It is necessary to find the probability distribution of demands number $i(t)$ in the recall source.

For probability distribution $P\ (k,\ n,\ i,\ t)$ of states $\{k,\ n,\ i\}$ considered RQ-system Kolmogorov's [1] differential equations is given by

$$\begin{cases}\dfrac{\partial P(0,n,i,t)}{\partial t}=-(\lambda_n+i\sigma)P(0,n,i,t)+\sum\limits_{\nu}P(0,\nu,i,t)q_{\nu n}+\mu P(1,n,i,t),\\ \dfrac{\partial P(1,n,i,t)}{\partial t}=-(\lambda_n+\mu)P(1,n,i,t)+\sum\limits_{\nu}P(1,\nu,i,t)q_{\nu n}+\lambda_n P(0,n,i,t)+\\ \qquad +\sigma(i+1)P(0,n,i+1,t)+\lambda_n P(1,n,i-1,t).\end{cases} \quad (1)$$

Method of the asymptotic semi-invariants

Applying system (1) for stationary distribution $P\ (k,\ n,\ i,\ t)\ =P\ (k,\ n,\ i)$, equate system defining characteristic functions [2]

$$H(k,n,u,t)=\sum_{i=0}^{\infty}e^{jui}P(k,n,i,t)=P\{k(t)=k,n(t)=n\}M\left\{e^{jui(t)}\middle|k(t)=k,n(t)=n\right\},$$

$$\begin{cases}-\sigma j\dfrac{\partial H(0,u)}{\partial u}=H(0,u)\{Q-\Lambda\}+\mu H(1,u),\\ \sigma j e^{-ju}\dfrac{\partial H(0,u)}{\partial u}=H(0,u)\Lambda+H(1,u)\left\{Q+\left(e^{ju}-1\right)\Lambda-\mu I\right\},\end{cases} \quad (2)$$

the solution $\{H\ (0,\ u),\ H\ (1,\ u)\}$ satisfies to normality condition

$$H(0,0)+H(1,0)=R\ ,$$

where Q is matrix of infinitesimal characteristics of Markov chain $k\ (t)$, Λ is the diagonal matrix with elements λ_n on the main diagonal, I is identity matrix, and row vectors

* Work is carried out at support of the analytical departmental target program: «Development of scientific potential of the higher school (2009-2010)» of Federal agency by education (project № 4781)

$$H(0,u) = \{H(0,1,u), H(0,2,u), \ldots, H(0,N,u)\},$$
$$H(1,u) = \{H(1,1,u), H(1,2,u), \ldots, H(1,N,u)\}.$$

For compact notation of the further calculations, we write down the system (2) in the following

$$\sigma j \frac{\partial H(u)}{\partial u} A(ju) = H(u) B(ju), \quad (3)$$

$$H(0)E = 1, \quad (4)$$

where E – identity row vector, and block matrixes A (ju) and B (ju) is given by

$$A(ju) = \begin{pmatrix} -I_N & \vdots & e^{-ju} I_N \\ \ldots & \vdots & \cdots \\ 0_N & \vdots & 0_N \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{(ju)^\nu}{\nu!} A_\nu, \quad B(ju) = \begin{pmatrix} Q - \Lambda & \vdots & \Lambda \\ \cdots & \vdots & \cdots \\ \mu I & \vdots & Q + (e^{ju} - 1)\Lambda - \mu I \end{pmatrix} = \sum_{\nu=0}^{\infty} \frac{(ju)^\nu}{\nu!} B_\nu .$$

The Asymptotic of first order

For a finding the asymptotic of first order denote σ = ε, and in the equation (3) realize replacements [3]

$$u = \varepsilon w, \quad H(u) = F_1(w, \varepsilon).$$

Then the equation (3) becomes

$$j \frac{\partial F_1(w,\varepsilon)}{\partial w} A(j\varepsilon w) = F_1(w,\varepsilon) B(j\varepsilon w), \quad (5)$$

and equality (4) we write down as follows

$$F_1(0,\varepsilon)E = 1. \quad (6)$$

In problem (5-6) we execute limiting transition at ε→ 0, get system

$$\begin{cases} j \dfrac{\partial F_1(w)}{\partial w} A_0 = F_1(w) B_0, \\ F_1(0)E = 1, \end{cases}$$

Solution F_1 (w) of this system we write down in the form of product

$$F_1(w) = R\Phi_1(w) = R \cdot \exp\{jw\kappa_1\}, \quad (7)$$

where vector R is defined by the system

$$\begin{cases} R(B_0 + \kappa_1 A_0) = 0, \\ RE = 1, \end{cases} \quad (8)$$

and $\Phi_1(w)$ is the scalar function. Values of κ_1 are defined as follows.

We combine all equations of system (5), postmultiplying this equation on identity column vector E, and get equality

$$j \frac{\partial F_1(w,\varepsilon)}{\partial w} A(j\varepsilon w)E = F_1(w,\varepsilon) B(j\varepsilon w)E,$$

in which matrixes are expanded

$$A(j\varepsilon w) = A_0 + j\varepsilon w A_1 + O(\varepsilon^2), \quad B(j\varepsilon w) = B_0 + j\varepsilon w B_1 + O(\varepsilon^2),$$

we get

$$j \frac{\partial F_1(w,\varepsilon)}{\partial w} j\varepsilon w A_1 E = F_1(w,\varepsilon) j\varepsilon w B_1 E + O(\varepsilon^2).$$

Limiting transition is realized here at ε→ 0 by substituting (7), we get the nonlinear scalar equation relative to κ_1

$$R(B_1 + \kappa_1 A_1)E = 0,$$

where vector $R = R(\kappa_1)$ is defined by system (8).

Function

$$h_1(u) = \exp\left\{ ju \frac{\kappa_1}{\sigma} \right\}$$

we will be called the asymptotic of the first order of characteristic function $H(u) = H(0, u) + H(1, u)$ the demands number $i(t)$ in a recall source, and value κ_1 / σ - the asymptotic semi-invariant of the first order.

The Asymptotic of second order

For a finding the asymptotic of second order in the equation (3) we realize the following replacement

$$H(u) = \exp\left\{ j\frac{u}{\sigma}\kappa_1 \right\} H_2(u).$$

Then for vector function $H_2(u)$ we get the equation

$$\sigma j \frac{\partial H_2(u)}{\partial u} A(ju) = H_2(u)\{B(ju) + \kappa_1 A(ju)\}, \qquad (9)$$

where solution $H_2(u)$ satisfies to the condition

$$H_2(0)E = 1. \qquad (10)$$

Now, in the system (9-10) denote $\sigma = \varepsilon^2$, and realize replacements

$$u = \varepsilon w, \quad H_2(u) = F_2(w, \varepsilon).$$

There have

$$j\varepsilon \frac{\partial F_2(w,\varepsilon)}{\partial w} A(j\varepsilon w) = F_2(w,\varepsilon)\{B(j\varepsilon w) + \kappa_1 A(j\varepsilon w)\},$$

$$F_2(0,\varepsilon)E = 1.$$

Further, we realize similar operations, as in the first asymptotic, we get the asymptotic of the second order of characteristic function $H(u)$ the demands number $i(t)$ in the recall source

$$h_2(u) = \exp\left\{ ju\frac{\kappa_1}{\sigma} + \frac{(ju)^2}{2}\frac{\kappa_2}{\sigma} \right\},$$

where

$$\kappa_2 = \frac{-\left\{ g_1(B_1 + \kappa_1 A_1)E + \frac{1}{2}R(B_2 + \kappa_1 A_2)E \right\}}{g(B_1 + \kappa_1 A_1)E + RA_1 E},$$

and value κ_2 / σ is the asymptotic semi-invariant of the second order and vectors g and g_1 are arbitrary particular solution of following equations systems

$$g(B_0 + \kappa_1 A_0) + RA_0 = 0,$$

$$g_1(B_0 + \kappa_1 A_0) + R(B_1 + \kappa_1 A_1) = 0.$$

Asymptotic some higher order

For a finding the asymptotic some higher order we apply a method of a mathematical induction [4].

Let vector-function $H_n(u)$ ($n \geq 3$) satisfies the equation

$$\sigma j \frac{\partial H_n(u)}{\partial u} A(ju) = H_n(u)\left\{ B(ju) + \kappa_1 A(ju) + \sum_{\nu=1}^{n-2} \frac{(ju)^\nu}{\nu!}\kappa_{\nu+1} A(ju) \right\}, \qquad (11)$$

in which all κ_ν are known by $\nu = 1,2,\ldots,n-1$.

Let applying the equation (11) value of κ_n has found, then in (11) we execute replacement

$$H_n(u) = \exp\left\{ \frac{(ju)^n}{n!}\frac{\kappa_n}{\sigma} \right\} H_{n+1}(u).$$

And get the equation for the vector-function $H_{n+1}(u)$

$$\sigma j \frac{\partial H_{n+1}(u)}{\partial u} A(ju) = H_{n+1}(u)\left\{ B(ju) + \kappa_1 A(ju) + \sum_{\nu=1}^{n-1} \frac{(ju)^\nu}{\nu!}\kappa_{\nu+1} A(ju) \right\}, \qquad (12)$$

solution $H_{n+1}(u)$ of this equation satisfies to a condition

$$H_{n+1}(0)E = 1 . \tag{13}$$

Further, applying the problem (12-13), we will find value of κ_{n+1}. For this in the problem (12-13) denote $\sigma = \varepsilon^{n+1}$, and execute replacements

$$u = \varepsilon w, \quad H_{n+1}(u) = F_{n+1}(w, \varepsilon),$$

we get

$$j\varepsilon^n \frac{\partial F_{n+1}(w,\varepsilon)}{\partial w} A(j\varepsilon w) = F_{n+1}(w,\varepsilon)\left\{ B(j\varepsilon w) + \kappa_1 A(j\varepsilon w) + \sum_{\nu=1}^{n-1} \frac{(j\varepsilon w)^\nu}{\nu!} \kappa_{\nu+1} A(j\varepsilon w) \right\},$$

$$F_{n+1}(0,\varepsilon)E = 1 .$$

Then, realizing similar operations, as in the first, and in the second asymptotics, we get the asymptotic of some higher order of characteristic function $H(u)$ the demands numbers $i(t)$ in the recall source

$$h_{n+1}(u) = \exp\left\{ \sum_{\nu=1}^{n+1} \frac{(ju)^\nu}{\nu!} \frac{\kappa_\nu}{\sigma} \right\},$$

where value κ_{n+1} is defined by equality

$$\kappa_{n+1} = -\left\{ g_n(B_1 + \kappa_1 A_1)E + \tfrac{1}{n+1} \sum_{\nu=1}^{n-1} C_{n+1}^{\nu} f_\nu \left(B_{m+1-\nu} + \sum_{k=0}^{n-\nu} C_{n+1-\nu}^{k} \kappa_{k+1} A_{n+1-\nu-k} \right) E + \right.$$
$$\left. + \tfrac{1}{n+1} R \left(B_{n+1} + \sum_{k=0}^{n-1} C_{n+1}^{k} \kappa_{k+1} A_{n+1-k} \right) E \right\} \Big/ \left\{ g(B_1 + \kappa_1 A_1)E + RA_1 E \right\},$$

and vectors g and g_n are defined by inhomogeneous systems of the linear algebraic equations

$$g(B_0 + \kappa_1 A_0) + RA_0 = 0,$$

$$g_n(B_0 + \kappa_1 A_0) + \sum_{\nu=1}^{n-1} C_n^{\nu} f_\nu \left(B_{n-\nu} + \sum_{k=0}^{n-\nu} C_{n-\nu}^{k} \kappa_{k+1} A_{n-\nu-k} \right) + R \left(B_n + \sum_{k=0}^{n-1} C_n^{k} \kappa_{k+1} A_{n-k} \right) = 0 .$$

and any additional conditions defining particular solution of these systems from the set of all their decisions, and vectors f_ν are defined by decomposition.

$$f_\nu = g_\nu + \kappa_{\nu+1} g , \ \nu = \overline{1, n-1} .$$

Thus, the research of RQ-systems by the method of asymptotic semi-invariant is carried out in this paper. Asymptotic probability distribution of the demands number in a recall source of some order is got.

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