INVESTIGATION OF THE QUEUEING SYSTEM $MMP^{(2)}/M/\infty$ BY METHOD OF THE MOMENTS*

Irina Ivanovskaya[1], Svetlana Moiseeva[2]

[1]The branch of Kemerovo State University in Anzhero-Sudzhensk,
Anzhero-Sudzhensk, Russia
[2]Tomsk State University, Tomsk, Russia
[1]*irinka_asf@mail.ru*, [2]*smoiseeva@mail.ru*

Productivity of the computer network is dependent on the period aspect operating When testing the productivity the duration of the computation process is the most important. The usage of the stochastic models is caused by the randomicity of the forming, calculating and transmitting processes. The stochastic models are the queueing systems and nets of different classes [1]. We build the mathematic models of the commutating networks and investigate the parameters of the real computation systems operating by the theory of queues [2].

Parallelism of information processing is one of the main principles in the modern network design [3, 4]. So analysis of the mathematical models with parallel operating service units and common arrival processes is of great practical importance.

We consider the queueing system that has two service units with unlimited number of servers. There exists arrival Markov-modulated Poisson process of double customers ($MMP^{(2)}$) where the matrix of infinitesimal characteristics is Q and the set of nonnegative numbers is λ_k, $k = 1,2,3,\dots K$ [5]. Two demands come simultaneously into the system at the moment events approach in considered arrival process.

One of the customers comes into the first unit, the other comes into the second one. Every customer occupy one of the free servers and is serviced during random time distributed with the exponential law where parameters are μ_1 and μ_2. This defines the dispatching rule.

We investigate two-dimensional stochastic process $\{n_1(t), n_2(t)\}$, which characterizes the number of busy servers in the first and the second service units at the moment t.

If the arrival process is not Poisson then the process $\{n_1(t), n_2(t)\}$ is not Markovian. But three-dimensional stochastic process $\{k(t), n_1(t), n_2(t)\}$ is the Markov chain. Where $k(t)$ - is the state of the controlling Markov chain so using investigation methods of the Markovian processes we define joint probability distribution

$$P(k,n_1,n_2,t) = P\{k(t)=k, n_1(t)=n_1, n_2(t)=n_2\}.$$

For probability distribution $P(k,n_1,n_2,t)$ we obtain the system of the differential Kolmogorov equations

$$\frac{\partial P(k,n_1,n_2,t)}{\partial t} = (-\lambda_k - n_1\mu_1 - n_2\mu_2 + q_{kk})P(k,n_1,n_2,t) + \lambda_k P(k,n_1-1,n_2-1,t) +$$
$$+(n_1+1)\mu_1 P(k,n_1+1,n_2,t) + (n_2+1)\mu_2 P(k,n_1,n_2+1,t) + \sum_{\nu\neq k} P(\nu,n_1,n_2,t)q_{\nu k}\,. \quad (1)$$

The initial conditions are $P(k,n_1,n_2,0) = P(k,0,0,t) = R(k)$, where $R(k)$ is the stationary probability distribution of the Markov chain states $k(t)$.

We solve the system (1) in steady-state conditions.

Consider the characteristic functions

* Work is carried out at support of the analytical departmental target program: «Development of scientific potential of the higher school (2009-2010)» of Federal agency by education (project № 4781)

$$H(k,u,w)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}e^{iun_1}e^{iwn_2}P(k,n_1,n_2),$$

where $i=\sqrt{-1}$ - is the imaginary value.
Consider, that

$$\frac{\partial H(k,u,w)}{\partial u}=i\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}n_1e^{iun_1}e^{iwn_2}P(k,n_1,n_2),$$

$$\frac{\partial H(k,u,w)}{\partial w}=i\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}n_2e^{iun_1}e^{iwn_2}P(k,n_1,n_2),$$

from (1) we get equations for $H(k,u,w)$:

$$\frac{\partial H(k,u,w)}{\partial t}=-\lambda_k H(k,u,w)+\mu_1 i\frac{\partial H(k,u,w)}{\partial u}+\mu_2 i\frac{\partial H(k,u,w)}{\partial w}+$$

$$+\lambda_k e^{i(u+w)}H(k,u,w)-\mu_1 ie^{-iu}\frac{\partial H(k,u,w)}{\partial u}-\mu_2 ie^{-iw}\frac{\partial H(k,u,w)}{\partial w}+\sum_{\nu}H(\nu,u,w)q_{\nu k}. \quad (2)$$

$$H(k,0,0)=R(k).$$

The system of the differential equations for characteristic functions $H(k,u,w)$ can be written by matrix:

$$\mu_1 i(e^{-iu}-1)\frac{\partial \mathbf{H}(u,w)}{\partial u}+\mu_2 i(e^{-iw}-1)\frac{\partial \mathbf{H}(u,w)}{\partial w}=\mathbf{H}(u,w)[\mathbf{\Lambda}(e^{i(w+u)}-1)+\mathbf{Q}], \quad (3)$$

where
$\mathbf{H}(u,w)=[\mathbf{H}(1,u,w),\mathbf{H}(2,u,w),\cdots]$,
$\mathbf{Q}=[q_{\nu k}]$ – is the matrix of infinitesimal characteristics,
$\mathbf{\Lambda}$ – is the diagonal matrix with elements λ_k on the main diagonal,
$\mathbf{E}$ - unit vector.

The moments of the first order

From the characteristic functions [7] we have

$$\left.\frac{\partial \mathrm{H}(k,u,w)}{\partial u}\right|_{\substack{w=0\\u=0}}=im_1(k),$$

$$\left.\frac{\partial \mathrm{H}(k,u,w)}{\partial w}\right|_{\substack{w=0\\u=0}}=im_2(k).$$

Consider $\overline{\mathbf{m}}_\mathbf{1}=[m_1(1),m_1(2),...]$, $\overline{\mathbf{m}}_2=[m_2(1),m_2(2),...]$, so
$\overline{\mathbf{m}}_\mathbf{1}\mathbf{E}=\sum_k m_1(k)=m_1$, $\overline{\mathbf{m}}_2\mathbf{E}=\sum_k m_2(k)=m_2$,

Where m_1, m_2 are the mathematical expectation of the customers number serving in every unit.

We differentiate equation (3) by variable u:

$$\mu_1 i(-i)e^{-iu}\frac{\partial \mathbf{H}(u,w)}{\partial u}+\mu_1 i(e^{-iu}-1)\frac{\partial^2 \mathbf{H}(u,w)}{\partial u^2}+\mu_2 i(e^{-iw}-1)\frac{\partial \mathbf{H}(u,w)}{\partial w}=$$

$$=\frac{\partial \mathbf{H}(u,w)}{\partial u}[\mathbf{\Lambda}(e^{i(w+u)}-1)+\mathbf{Q}]+\mathbf{H}(u,w)[\mathbf{\Lambda}e^{i(w+u)}i]. \quad (4)$$

So if $u=w=0$ then vector $\overline{\mathbf{m}}_\mathbf{1}$ is

$$\mu_1\overline{\mathbf{m}}_\mathbf{1}=\overline{\mathbf{m}}_\mathbf{1}\mathbf{Q}+\mathbf{R\Lambda}, \quad (5)$$

and

$$\overline{\mathbf{m}}_1 = \mathbf{R\Lambda}\left[\mu_1\mathbf{I} - \mathbf{Q}\right]^{-1}. \qquad (6)$$

Summing up all the equations of system (5), we get the average number of busy servers in the first service unit

$$m_1 = \frac{1}{\mu_1}\mathbf{R\Lambda}\,\mathbf{E}.$$

In the same way we find the moment of the first order for the second service unit by the differentiating equation (3) with variable w. Having $u = w = 0$, we find vector-line

$$\overline{\mathbf{m}}_2 = \mathbf{R\Lambda}\left[\mu_2\mathbf{I} - \mathbf{Q}\right]^{-1}, \qquad (7)$$

and the average number of busy servers in the second service unit

$$m_2 = \frac{1}{\mu_2}\mathbf{R\Lambda}\,\mathbf{E}.$$

The moment of the second order

We find the initial second order moments of the busy servers count in the first service unit by differentiating equation (4) with u

$$\mu_1 i\frac{\partial\mathbf{H}(u,w)}{\partial u} + 2\mu_1 i(-i)e^{-iu}\frac{\partial^2\mathbf{H}(u,w)}{\partial u^2} + \mu_1 i(e^{-iu} - i)\frac{\partial^3\mathbf{H}(u,w)}{\partial u^3} =$$

$$= \frac{\partial^2\mathbf{H}(u,w)}{\partial u^2}[\mathbf{\Lambda}(e^{i(w+u)} - 1) + \mathbf{Q}] + 2\frac{\partial\mathbf{H}(u,w)}{\partial u}\mathbf{\Lambda}ie^{i(w+u)} + \mathbf{H}(u,w)[\mathbf{\Lambda}e^{i(w+u)}i^2]. \qquad (8)$$

Note that

$$\left.\frac{\partial^2\,\mathrm{H}(k,u,w)}{\partial u^2}\right|_{\substack{w=0\\u=0}} = i^2 m_1{}^2(k),$$

$$\left.\frac{\partial^2\,\mathrm{H}(k,u,w)}{\partial w^2}\right|_{\substack{w=0\\u=0}} = i^2 m_2{}^2(k),$$

we consider $\overline{\mathbf{m}}_1^2 = \left[m_1^2(1), m_1^2(2),...\right]$, $\overline{\mathbf{m}}_2^2 = \left[m_2^2(1), m_2^2(2),...\right]$ - vector lines

$$\overline{\mathbf{m}}_1^2\mathbf{E} = \sum_k m_1^2(k) = m_1^2,$$

$$\overline{\mathbf{m}}_2^2\mathbf{E} = \sum_k m_2^2(k) = m_2^2,$$

where m_1^2, m_2^2 – are the second moments of the customers number serving in every unit.
We get an equation for the moment of the second order for the first service unit

$$\overline{\mathbf{m}}_1^2\left[2\mu_1\mathbf{I} - \mathbf{Q}\right] = \overline{\mathbf{m}}_1\left[\mu_1\mathbf{I} - 2\mathbf{\Lambda}\right] + \mathbf{R\Lambda}.$$

Substituting for (5) we get

$$\overline{\mathbf{m}}_1^2 = \mathbf{R\Lambda}\left\{\mathbf{I} + \left[\mu_1\mathbf{I} - \mathbf{Q}\right]^{-1}\left[\mu_1\mathbf{I} - 2\mathbf{\Lambda}\right]\right\}\left[2\mu_1\mathbf{I} - \mathbf{Q}\right]^{-1}\mathbf{E},$$

Similarly for the second service unit

$$\overline{\mathbf{m}}_2^2 = \mathbf{R\Lambda}\left\{\mathbf{I} + \left[\mu_2\mathbf{I} - \mathbf{Q}\right]^{-1}\left[\mu_2\mathbf{I} - 2\mathbf{\Lambda}\right]\right\}\left[2\mu_2\mathbf{I} - \mathbf{Q}\right]^{-1}\mathbf{E}.$$

We get the correlation moment for two-dimensional stochastic process $\{n_1(t), n_2(t)\}$ by differentiating equation (4) with variable w

$$\mu_1 i(-i)e^{-iu}\frac{\partial\mathbf{H}^2(u,w)}{\partial u\partial w} + \mu_1(-i^2)e^{-iu}\frac{\partial\mathbf{H}(u,w)}{\partial u} + \mu_1 i(e^{-iu} - 1)\frac{\partial^3\mathbf{H}(u,w)}{\partial u^2\partial w} +$$

$$+\mu_2 i(e^{-iw}-1)\frac{\partial \mathbf{H}^2(u,w)}{\partial^2 w}+\mu_2(-i^2)e^{-iw}\frac{\partial \mathbf{H}(u,w)}{\partial w}=\frac{\partial^2 \mathbf{H}(u,w)}{\partial u \partial w}[\mathbf{\Lambda}(e^{i(w+u)}-1)+\mathbf{Q}]+$$

$$+\frac{\partial \mathbf{H}(u,w)}{\partial u}[\mathbf{\Lambda} i e^{i(w+u)}]+\frac{\partial \mathbf{H}(u,w)}{\partial w}[\mathbf{\Lambda} i e^{i(w+u)}]+\mathbf{H}(u,w)[\mathbf{\Lambda} e^{i(w+u)} i^2]. \quad (9)$$

For the characteristic functions there exists

$$\left.\frac{\partial^2 \mathrm{H}(k,u,w)}{\partial u^2}\right|_{\substack{w=0\\u=0}}=\left.\sum_{n_1}\sum_{n_2} i^2 n_1 n_2 e^{iun_1} e^{iwn_2} P(k,n_1,n_2)\right|_{\substack{w=0\\u=0}}=i^2 m(k). \quad (10)$$

Denote $\overline{\mathbf{m}}=[m(1),m(2)\ldots]$, then $\overline{\mathbf{m}}\mathbf{E}=\sum_k m(k)=m$.

Substituting $u=w=0$ in (9) and taking into account (10), we obtain heterogeneous system of linear algebraic equation

$$\overline{\mathbf{m}}[(\mu_1+\mu_2)\mathbf{I}-\mathbf{Q}]=(\overline{\mathbf{m}}_1+\overline{\mathbf{m}}_2)\mathbf{\Lambda}+\mathbf{R\Lambda}.$$

Summing up all the equations of the system we get the correlation moment of the two-dimensional stochastic process $\{n_1(t),n_2(t)\}$

$$m=\frac{1}{\mu_1+\mu_2}\left(\mathbf{R\Lambda E}+(\overline{\mathbf{m}}_1+\overline{\mathbf{m}}_2)\mathbf{\Lambda E}\right),$$

where vector-lines $\overline{\mathbf{m}}_1$, $\overline{\mathbf{m}}_2$ have been already defined.

The average workload of computing devices can be defined by the determined stochastic characteristics. Therefore the productivity of the computation system can be further estimated.

References

1. Tsoy S.A Characteristic function appliance in asymptotic transmission network investigations with statistical protocols of access random set. // Bulletin TSU. –Tomsk, 2006. –№293 . - PP.129-134.
2. Nazarov A.A., Semenova I.A., Comparison of asymptotic and prelimit results in analysis of RQ-system// Collected science articles. –Minsk, 2010.Number 3. PP.272-277.
3. Toporkov V.V. Models of grid computing. –M.: FIZMATLIT, 2004. – 320 p.
4. Endrus G.R Basics of multithreaded, parallel and distributed software technology: Transl. from English. – M.: Publishing house «Williams», 2003. – 512 p.
5. Nazarov A.A., Terpugov A.F. The queueing system. Tomsk : Publishing house NTL, 2005. – 228 p.
6. Gnedenko B.V., Kovalenko I.N. Introduction in theory of queues. Publication 3, correct and enriched.M.: KomKniga, 2005. – 408 p.
7. Nazarov A.A., Terpugov A.F. Theory of chances and stochastic process. Tomsk: Publishing house NTL, 2006. – 204 p.
8. Nazarov A.A., Moiseeva S.P. Method of asymptotic analysis in theory of queuing systems. – Tomsk: NTL, 2006. – 112 p.