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STOCHASTIC DERIVATIVE OPERATOR OF TWO-DIMENSIONAL POISSON FUNCTIONALS

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0. In the theory of stochastic Ito's integral $\int_0^T f(t,\omega)dw_t$, besides the fact that the integrand $f(t,\omega)$ is the measurable function of two variables, it should be the adapted (nonanticipated) process. Starting from the 70th of the past century, many attempts were made to weak the requirement for the integrand to be adapted for the integrand of the Ito's stochastic integral as well as in the theory of "the extension of filtration". Skorokhod (1975) suggested absolutely different method, it generalized the direct and inverse Ito's integrals and did not require for the integrand to be independent of the future Wiener process. Towards this end, he required for the integrand to be smooth in a certain sense, i.e., its stochastic differentiability. This idea was later on developed in the works of Gaveau-Trauber (1982), Nualart, Zakai (1986), Pardoux (1982), Protter, Malliavin (1979), etc. In particular, Gaveau-Trauber have proved that the Skorokhod operator of stochastic integration coincides with the conjugate operator of a stochastic derivative operator.

For the class of normal martingales (a martingale M is called normal if $\langle M,M\rangle_t = t$) which have the chaos representation property Ma, Protter and Martin (1998) have proposed an anticipating integral and the stochastic derivative operator and the integral representation formula of Ocone-Haussmann-Clark is established (which, in turn play an important role in the modern financial mathematics). This integral is analogous to the Skorohod integral as developed by Nualart and Pardoux (1988). According to the Ocone-Haussmann-Clark formula if $F \in D_{2,1}^M$, then

$$F = E(F) + \int_0^T {}^p(D_t^M F)dM_t$$

is valid; here $D_{2,1}^M$ denotes the space of quadratically integrable functionals having the first order stochastic derivative, and ${}^p(D_t^M F)$ is the predictable projection of the stochastic derivative $D_t^M F$ of the functional F. There are many similarities between the above-mentioned martingale anticipating integral and the Skorohod integral, but there are also some important differences. Many of these differences stem from one key fact: in the Wiener case $[w,w]_t = \langle w,w\rangle_t = t$, while in the normal martingale case only $\langle M,M\rangle_t = t$, and $[M,M]_t$ is random. For example, there are two ways to describe the variational derivative and they are equivalent in the Wiener case but not in the martingale case. In [3] an example is given, which shows that the two definitions (Sobolev space and chaos expansion) are compatible if and only if $[M,M]_t$ is deterministic. Therefore in the martingale case the space $D_{p,1}^M$ $(1 < p < 2)$ cannot be defined in the usual way, i.e., by closing the class of smooth functionals with respect to the corresponding norm. In work of Purtukhia (2003) the space $D_{p,1}^M$ $(1 < p < 2)$ is proposed for a class of normal martingales and the integral representation formula of Ocone-Haussmann-Clark is established for functionals from this space.

1. Let $w_t, t \in [0,1]$ be a d-dimensional standard Wiener process defined on the canonical probability space $(\Omega, \Im, P)$, $\Im_t = \sigma\{w_s, 0 \le s \le t\}$. A smooth functional will be a random variable $F:\Omega \to R^1$ of the form $F = f(w_{t_1}, w_{t_2}, ..., w_{t_n})$, where the function f belongs to $C_b^\infty(R^{dn})$ and $t_1, t_2, ..., t_n \in [0,1]$. The derivative of F can be defined as (see [2]):

$$(D_t^w F)^j = \sum_{i=1}^n \frac{\partial f}{\partial x^{ji}}(w_{t_1}, w_{t_2}, ..., w_{t_n}) I_{[0,t_i]}(t),\ t \in [0,1],\ j = 1,...,d.$$

Let F be a square integrable random variable having an orthogonal Wiener-Chaos expansion of the form $F = \sum_{n=0}^\infty I_n(f_n)$. Then F belongs to the space $D_{2,1}^w$ (see [2]) if and only if $\sum_{n=1}^\infty nn! \| f_n \|^2_{L_2([0,1]^n)} < \infty$ and in this case we have $D_t^w F = \sum_{n=1}^\infty n I_{n-1}(f_n(\cdot,t)),\quad t \in [0,1]$ and

$$\|\|\| D_\cdot^w F \|_{L_2([0,1])} \|_{L_2(\Omega)} = \sum_{n=1}^\infty nn! \| f_n \|^2_{L_2([0,1]^n)}.$$

2. Let Σ_n be an increasing simplex of R_+^n: $\Sigma_n = \{(t_1,...,t_n) \in R_+^n : 0 < t_1 < \cdots < t_n\}$, and extend a function f defined on Σ_n by making f symmetric on R_+^n. One can then define the multiple integral with respect to M as

$$I_n(f) := n! \int_{\Sigma_n} f(t_1,...,t_n) dM_{t_1} \cdots dM_{t_n}.$$

Definition 2.1 (cf. Definition 3.2 [3]). Let $\Re = \sigma\{M_t; t \ge 0\}$ be the σ-algebra generated by a normal martingale M. Let H_n be the n-th homogeneous chaos, $H_n = I_n(f)$, where f ranges over all $L_2(\Sigma_n)$. If $L_2(\Re, P) = \bigoplus_{n=0}^\infty H_n$, then we say M possesses the chaos representation property (CRP).

Let $(\Omega, \Im, \{\Im_t\}_{t\ge0}, P)$ be a filtered probability space satisfying the usual conditions. We assume that a normal martingale M with the CRP is given on it and that $\Im$ is generated by M. Thus, for any random variable $F \in L_2(\Re, P)$ we have by the CRP that there exists a sequence of functions $f_n \in L_s^2([0,1]^n)$ $(=\{h \in L_2([0,1]^n):\ h$ is symmetric in all variables), $n = 1,2,...$, such that $F = \sum_{n=0}^\infty I_n(f_n)$. Consider the following subset $D_{2,1}^M \subset L_2(\Re, P)$:

$$D_{2,1}^M = \{F = \sum_{n=0}^\infty I_n(f_n)\ : \sum_{n=1}^\infty nn! \| f \|^2_{L_2([0,1]^n)} < \infty\}.$$

Definition 2.2 (see [3]). The derivative operator is defined as a linear operator $D_\cdot^M$ from $D_{2,1}^M$ into $L^2([0,T]\times\Omega)$ by the relation:

$$D_t^M F := \sum_{n=1}^\infty n I_{n-1}(f_n(\cdot,t)),\quad t \in [0,1],$$

whenever F has the chaos expansion $F = \sum_{n=0}^\infty I_n(f_n)$.

3. Our aim is to introduce a new definition of the stochastic derivative operator for the two-dimensional compensated Poisson functionals, which is not based on the chaos expansion of functionals, as well as in Ma, Protter and Martin's work and to show the equivalence of this two definitions. Let $(\Omega, \Im, \{\Im_t\}_{t\in[0.T]}, P)$ be a filtered probability space satisfying the usual conditions. Let N_t be the standard Poisson process ($\mathrm{P}(N_t = k) = t^k e^{-t}/k!, k = 0,1,2,...$) and $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Let M_t be the compensated Poisson process ($M_t = N_t - t$). Let us denote $\nabla_x f(x) := f(x+1) - f(x)$; $\nabla_x f(M_T) := \nabla_x f(x)\big|_{x=M_T}$. For any function of two variables $g(\cdot,\cdot)$ introduce the designation: $\nabla^2 g(x,y) = g(x+1, y+1) - g(x,y)$.

It is not difficult to see that $\nabla_x[\nabla_y g(x,y)] = \nabla_y[\nabla_x g(x,y)]$ and

$$\nabla^2 g(x,y) = \nabla_x[\nabla_y g(x,y)] + \nabla_x g(x,y) + \nabla_y g(x,y).$$

Using the relations

$$M_s = \int_0^T I_{[0,s]}(u)dM_u = I_1(I_{[0,s]}(\cdot)) \text{ and } [M,M]_s = N_s = M_s + s,$$

by the Definition 2.2 we can obtain: $D_t^M M_s = D_t^M[I_1(I_{[0,s]}(\cdot))] = I_{[0,s]}(t)$ and

$$D_t^M[M,M]_s = D_t^M N_s = D_t^M M_s + D_t^M s = I_{[0,s]}(t).$$

Definition 3.1. $\overline{D}_t^M (M_s)^n := [\nabla_x(x^n)]|_{x=M_s} \cdot \overline{D}_t^M M_s := [\nabla_x(x^n)]|_{x=M_s} \cdot I_{[0,s]}(t)$;

$$\overline{D}_t^M P(M_S, M_T) = \nabla_y \nabla_x P(M_S, M_T) I_{[0,S]}(t) I_{[0,T]}(t) + $$
$$+ \nabla_x P(M_S, M_T) I_{[0,S]}(t) + \nabla_y P(M_S, M_T) I_{[0,T]}(t),$$

for any polynomial function $P(x,y)$.

Proposition 3.1. If $F = I_2(f_2)$ for some $f_2 \in L_s^2([0,T]^2)$, then F have the stochastic derivative, $\overline{D}_t^M F = 2I_1(f_2(\cdot,t)) = D_t^M F$ and

$$\| \overline{D}_t^M F \|^2_{L_2([0,T]\times\Omega)} = 2\cdot 2! \cdot \| f_2 \|^2_{L_2([0,T]^2)}.$$

Proof. Step 1:Suppose that f_2 is a symmetric function of the form $f_2(t_1,t_2) = aI_{A_1\times A_2}(t_1,t_2) + aI_{A_2\times A_1}(t_1,t_2)$, where $A_1, A_2 \subset [0,T]$, $A_1 \cap A_2 = \varnothing$. The set of such symmetric function we denote by E_2. For such f_2 we have

$$I_2(f_2) = a\int_0^T I_{A_1}(s)dM_s \int_0^T I_{A_2}(s)dM_s + a\int_0^T I_{A_2}(s)dM_s \int_0^T I_{A_1}(s)dM_s = 2aM(A_1)M(A_2)$$

Therefore, due to the Definition 3.1, one can easily verify that:

$$\overline{D}_t^M I_2(f_2) = 2a\overline{D}_t^M[M(A_1)M(A_2)] = 2a[I_{A_1}(t)I_{A_2}(t) + I_{A_1}(t)M(A_2) + I_{A_2}(t)M(A_1)] =$$
$$= 2a[I_{A_1}(t)M(A_2) + I_{A_2}(t)M(A_1)] = 2I_1(f_2(\cdot,t)). \quad (3.1)$$

Moreover, it is not difficult to see that:

$$\| \overline{D}_t^M F \|^2_{L_2([0,T]\times\Omega)} = \int_o^T \| 2I_1(f_2(\cdot,t)) \|^2_{L_2(\Omega)} dt = \int_o^T 2^2 \cdot 1! \cdot \| I_1(f_2(\cdot,t)) \|^2_{L_2([0,T])} dt =$$
$$= 2\cdot 2! \cdot \int_o^T \| I_1(f_2(\cdot,t)) \|^2_{L_2([0,T])} dt = 2\cdot 2! \cdot \| f_2 \|^2_{L_2([0,T]^2)}. \quad (3.2)$$

Step 2: If $F = I_2(f_2)$ for some $f_2 \in L_s^2([0,T]^2)$, then F can be approximated in the $L_2(\Omega)$ -norm by a sequence of multiple integrals $I_2(f_2^n)$ of elements $f_2^n \in \mathrm{E}_2$ as $n \to \infty$. By the relations (3.1) and (3.2) applied to f_2^n we deduce that the sequence of derivatives $D_{\cdot}^M f_2^n$ converge in $L_2([0,T]\times\Omega)$, which completes the proof of the proposition.

Analogously one can prove the following

Theorem 3.1. For two-dimensional Poisson polynomial functionals the above-given two definitions of stochastic derivatives (Definition 3. 2 from [3] and Definition 3.1) are equivalent:

$$\overline{D}_t^M P(M_S, M_T) = D_t^M P(M_S, M_T).$$

Proposition 3.2. $D_t P(M_S, M_T) = [P(M_S+1, M_T+1) - P(M_S, M_T+1)] I_{[0,S]}(t) +$
$+ [P(M_S, M_T+1) - P(M_S, M_T)] I_{[0,T]}(t) = \nabla_x P(M_S, M_T+1) D_t M_S + \nabla_y P(M_S, M_T) D_t M_T$.

Proposition 3.3. For any polynomial functions $F(x, y)$ and $G(x, y)$ we have

$$D_t[F(M_S, M_T)G(M_S, M_T)] = G(M_S, M_T)D_t F(M_S, M_T) +$$
$$+ F(M_S, M_T)D_t G(M_S, M_T) + D_t F(M_S, M_T)D_t G(M_S, M_T).$$

Proof. Due to the definition 3.1 on the one hand we have

$$D_t[F(M_S, M_T)G(M_S, M_T)] = [F(M_S+1, M_T+1)G(M_S+1, M_T+1) -$$
$$- F(M_S, M_T+1)G(M_S, M_T+1)] I_{[0,S]}(t) + [F(M_S, M_T+1)G(M_S, M_T+1) -$$
$$- F(M_S, M_T)G(M_S, M_T)] I_{[0,T]}(t) := I_1 + I_2.$$

On other hand, one can conclude that

$$G(M_S, M_T)D_t F(M_S, M_T) + F(M_S, M_T)D_t G(M_S, M_T) +$$
$$+ D_t F(M_S, M_T)D_t G(M_S, M_T) = I_1 + I_2.$$

Theorem 3.2. Let u_t is Skorokhod integrable and $F(x, y)$ is a polynomial function. Then $F(M_S, M_T)u_t$ is Skorokhod integrable and we have

$$\int_{[0,T]} F(M_S, M_T)u_t dM_t = F(M_S, M_T)\int_{[0,T]} u_t dM_t -$$
$$- \int_{[0,T]} u_t D_t[F(M_S, M_T)]dM_t - \int_{[0,T]} u_t D_t[F(M_S, M_T)]dt \quad (P-a.s.).$$

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