LOCALLY-DIFFERENTIAL ANALOGUE OF THE BASIC LEMMA OF THE GALTON-WATSON PROCESSES AND THE Q-PROCESSES

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Consider the following evolution scheme of some population of particles. Let random variables $\{Z_n,\ n\in\mathbf{N}_0\}$ ($\mathbf{N}_0=\{0\}\bigcup\{\mathbf{N}=1,2,...\}$) is given recursively by

$$Z_0=1,\quad Z_{n+1}=\sum_{k=1}^{Z_n}\xi_{nk}\ .$$

Here the independent and identically distributed random variables ξ_{nk} is interpreted as the number of offspring of the k th individual in the n th generation. All offspring variables ξ_{nk} have a common distribution for all $n,k\in\mathbf{N}$. Then Z_n be viewed as the population size at time n in Galton-Watson Branching Process (GWP). The value $A:=\mathbf{E}\xi_{nk}$ denotes the mean number of offspring of a single individual. Further we consider the case of $A=1$, at which the GWP $\{Z_n,\ n\in\mathbf{N}_0\}$, according to classification of branching processes, is called *critical*.

Let $p_k:=\mathbf{P}\{Z_1\equiv\xi_{01}=k\}$ be are reproduction law of offspring of the single individual, for which we everywhere demand a conditions $p_0>0$ and $p_0+p_1\neq 1$. Put into consideration the probability generating function (g.f.) $F(x):=\sum_{k\in\mathbf{N}_0}p_kx^k$. According to a branching condition, the g.f. $F_n(x):=\mathbf{E}x^{Z_n}$ of the variable Z_n is defined by n step iteration of $F(x)$, that is for any $n,m\in\mathbf{N}$ the relations $F_{n+m}(x)=F_n(F_m(x)),\ F_0(x)=x$ hold; see, e.g. [1, p.2]. Let's assume further, that the second moment $B:=F''(1)$ is finite.

It is known, that asymptotical behavior of function $R_n(x)=1-F_n(x)$ play a special role in researches of the trajectories of critical GWP. The following statement holds.

Lemma A [1, p.19]. *If $A=1$, then for all $0\le x<1$ following asymptotical representation is fair:*

$$R_n(x)=\frac{1-x}{\frac{Bn}{2}(1-x)+1}\big(1+o(1)\big),\ n\to\infty\,. \tag{1}$$

Due to its importance, last lemma is called the basic lemma of the theory of critical GWP.

At $x=0$ the value $R_n(0)=\mathbf{P}\{Z_n>0\}$ represents the survival probability of GWP $\{Z_n,\ n\in\mathbf{N}_0\}$. This probability tends to zero by the order $O\left(1/n\right)$ at infinite growth of number of generations n, i.e. the critical GWP asymptotically generates. Therefore in this case the properties of trajectories of GWP are traditionally studied on non-zero trajectories. Thus the important role is played by g.f.

$$g_n(x):=\sum_{j\in\mathbf{N}}\mathbf{P}\left\{Z_n=j\,\middle|\,Z_n>0\right\}x^j=1-\frac{R_n(x)}{R_n(0)}\,. \tag{2}$$

An important value represents and an asymptotical representation of function $R_n'(x)$ as $n\to\infty$. We have found out this representation the neighborhood of point $x=1$. The latter remark associates on the one hand with difficulty of receipt of representation for $0 \le x < 1$, on the other hand it sufficient for our further discussing. So, the following locally-differential analog of the basic lemma of the theory of critical GWP is fair.

Lemma 1. *If $A = 1$, then as $x \to 1$ following asymptotical representation is fair:*

$$R_n'(x) \sim -\, g_n^2(x), \;\; n \to \infty, \tag{3}$$

where the g.f. $g_n(x)$ is defined by (2).

Proof. As the second moment $B := F''(1)$ is finite, the Taylor expansion gives the chance to write to us that

$$F(x) = x + \frac{B}{2}(x-1)^2 \left(1 + o(x-1)^2\right), \;\; x \to 1. \tag{4}$$

Whence by iteration of $F_n(x)$ it follows

$$F_n(F(x)) - F_n(x) = \frac{B}{2} R_n^2(x)\left(1 + o(1)\right), \;\; n \to \infty. \tag{5}$$

Using the Lagrange theorem in the left part of (5) we have

$$F_n'\left(c(x)\right) = \frac{B}{2(F(x)-x)} R_n^2(x)\left(1 + o(1)\right), \;\; n \to \infty, \tag{6}$$

where $c(x) = x + (F(x)-x)\theta$, $0 < \theta < 1$. In turn, owing to the relation (4) we will be convinced that $c(x) \sim x$, $x \to 1$. Considering last fact together with formulas (4), (6), and taking into account a continuity of derivative of g.f. we will receive as $x \to 1$

$$F_n'(x) \sim \left[\frac{1}{1-x} R_n(x)\right]^2, \;\; n \to \infty. \tag{7}$$

Combining (1), (2) and (7), we complete the proof.

The continuous time analogue of the last lemma has been proved in work of the author [2]. There some are resulted application of this lemma for the Markov Branching Processes.

Remark. As the simple appendix of the lemma 1 we may to result its application in the proof of classical Yaglom's theorem, which confirms, that the random variable $2Z_n/Bn$ converges in weakly to a random variable distributed by the exponential law; see. [1, c.20]. *Really*, the Laplace transform (LT) $\varphi_n(\theta) := \mathbf{E}\left[e^{-2\theta Z_n/Bn} \middle| Z_n > 0\right]$ we write down in the form of $\varphi_n(\theta) = g_n(\theta_n)$ and, after differentiating it, taking into account (2) and (3), we receive

$$\varphi_n'(\theta) \sim -\, g_n^2(\theta_n) = -\varphi_n^2(\theta), \;\; n \to \infty, \tag{8}$$

where $\theta_n := \exp\{-2\theta/Bn\}$, $\theta > 0$. As the LT of exponential law is the solution of differential equation

$$\varphi'(\theta) + \varphi^2(\theta) = 0,$$

with the initial condition $\varphi(0) = 1$, then according to ideas of work [2], the equation (8) confirms that

$$\varphi_n(\theta) \to \frac{1}{1+\theta}, \;\; n \to \infty.$$

The last convergence is equivalent to the statement of Yaglom's theorem.

In the present paper we are discussing some applications of the lemma 1 in researches of asymptotic properties of Q-processes.

The Q-process is the homogeneous Markov chain $\left\{W_n, \, n \in \mathbf{N}_0\right\}$ with initial state $W_0 = 1$, which is defined by transition probabilities

$$Q_{ij}^{(n)} := \mathbf{P}\left\{W_{n+k} = j \mid W_k = i\right\} = \lim_{m\to\infty} \mathbf{P}\left\{Z_{n+k} = j \mid Z_k = i,\ Z_{n+k+m} > 0\right\}$$

for $n, i, j, k \in \mathbf{N}$. After calculation we will be convinced that

$$Q_{ij}^{(n)} = \frac{j}{iA^n} \mathbf{P}\left\{Z_{n+k} = j \mid Z_k = i\right\}; \tag{9}$$

on details see [1, pp. 56-58]. Further we need the g.f.

$$W_n^{(i)}(x) := \sum_{j\in\mathbf{N}_0} Q_{ij}^{(n)} x^j .$$

From equality (9) and taking into account the iteration for g.f. $F_n(x)$, we will receive that

$$W_n^{(i)}(x) = \left[F_n(x)\right]^{i-1} W_n(x),$$

where g.f. $W_n(x) := W_n^{(1)}(x) = \mathbf{E}\left[x^{W_n} \middle| W_0 = 1\right]$ is form of

$$W_n(x) = -xR_n'(x),\ n \in \mathbf{N}, \tag{10}$$

Further discussion gives to us that the following limit exists:

$$\lim_{n\to\infty} n^2 W_n^{(i)}(x) = \lim_{n\to\infty} n^2 W_n(x) =: \mu(x), \tag{11}$$

and limit g.f. $\mu(x) = \sum_{k\in\mathbf{N}} \mu_k x^k$ satisfies the functional equation

$$W_1(x)\mu(F(x)) = F(x)\mu(x).$$

Besides the non-negative numbers $\left\{\mu_n,\ n \in \mathbf{N}\right\}$ form a stationary measure for Q-processes. Moreover $\sum_{j\in\mathbf{N}} \mu_j = \infty$, and

$$n^2 Q_{ij}^{(n)} = \mu_j\left(1 + o(1)\right),\ n \to \infty. \tag{12}$$

Theorem 1. *Let $A = 1$ and the stationary measure $\left\{\mu_n,\ n \in \mathbf{N}\right\}$ of Q-process is given by (12). Then*

$$\lim_{n\to\infty} \frac{1}{n^2}\left[\mu_1 + \mu_2 + \ldots + \mu_n\right] = \frac{2}{B^2}. \tag{13}$$

Proof. By using (1) and (2), the formula (3) we transform to a kind of

$$R_n'\left(x\right) \sim -\frac{4}{B^2 n^2} \frac{1}{(1-x)^2},\ n \to \infty,$$

as $x \to 1$. Considering equalities (10) and (11), from last relation we will receive that

$$\mu\left(x\right) \sim \frac{4}{B^2} \frac{1}{(1-x)^2},\ x \to 1. \tag{14}$$

Now we are in conditions of well-known Hardy-Littlewood Tauberian theorem, according to which each of relations (13) and (14) attract another.

The theorem is proved.

The statement of the lemma 1 much more simplifies the proof of the following theorem, observed by T.Harris in 1951.

Theorem 2 [1, p.59]. *Let $A = 1$. Then for any $x > 0$*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{W_n}{\mathbf{E}W_n} \le x\right\} = 1 - e^{-2x} + 2xe^{-2x}.$$

Proof. Consider LT $\psi_n(\theta) := \mathbf{E}\left[e^{-\theta W_n/\mathbf{E}W_n}\right]$ of the variable $W_n / \mathbf{E}W_n$ and in view of equality (10), we will write down it in a form of

$$\psi_n(\theta) = -e^{-\theta/\mathbf{E}W_n} R_n\left(e^{-\theta/\mathbf{E}W_n}\right).$$

By means of (10) we can calculate, that

$$\mathbf{E}W_n = W_n'(1) = Bn + 1 .$$

Considering last expression and owing to relations (3) and (8) we will have

$$\psi_n(\theta) \sim \varphi_n^2\left(\frac{\theta}{2}\right), \ n \to \infty .$$

We have noticed in remark, that $\varphi_n(\theta) \to 1/[1+\theta]$ as $n \to \infty$. Hence, we conclude, that

$$\psi_n(\theta) \to \frac{1}{\left[1+\frac{\theta}{2}\right]^2}, \ n \to \infty .$$

Received LT corresponds to the Erlang's density $4xe^{-2x}$, $x \geq 0$ of the first order, received by compositions of two exponential laws with identical parameter $\lambda = 2$. It is equivalent to statement of the theorem.

We notice that the theorem 2 in the monograph [1] has been proved by means of a consequence of Helly's theorem.

References

[1] K.B. Athreya, P.E. Ney, *Branching processes*, Springer, New York, **1972**.
[2] A.A. Imomov, *A Differential Analog of the Main Lemma of the Theory of Markov Branching Processes and Its Applications*, Ukrainian Mathematical Journal, **7** (**2005**), **2**, Springer, New York, pp. 307-315.