# ON ONE CRITERION OF CONVERGENCE TO THE EXPONENTIAL LAW

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In modern researches in proof of limit theorems, besides the continuity theorem, characteristic properties of distribution laws are used. When these properties represent the differential equation which satisfies characteristic function or Laplace transform (and generating function also) of limit distribution law, the theorem statement consists in stability of solution of this equation. In normal approximation case such approach to problem is formed the Stein–Tikhomirov method; see [1]. In paper [2] is offered the simplified variant of Stein–Tikhomirov method, realized by definition of one differential operator for characteristic function. By means of this method the criterion of justice of the non-classical Central Limit Theorem has been established.

In present report, being based on ideas of work [2], the new criterion of justice of the theorem of convergence to exponential law will be established.

We clap eyes on, that the received criterion may be useful in research of Stochastic Branching Processes.

Let

$$\Gamma_\alpha(x) = \begin{cases} 1 - e^{-\alpha x}, & x \ge 0 \\ 0 \quad , & x < 0 \end{cases},$$

$\alpha > 0$, is the distribution function of exponential law.

Following Formanov [2], we consider Laplace transform class $\psi(\theta)$ by follows:

$$\Phi = \left\{ \psi(\theta) \colon \left|\psi'(0)\right| = \frac{1}{\alpha} \right\}.$$

In class $\Phi$ we will enter into consideration the operator

$$\Delta\big(\psi(\theta)\big) := \psi'(\theta) + \frac{1}{\alpha}\psi^2(\theta). \tag{1}$$

Since the Laplace transform of exponential law

$$\psi_\alpha(\theta) = \int_0^\infty e^{-\theta x} d\Gamma_\alpha(x)$$

is

$$\psi_\alpha(\theta) = \frac{1}{\left[1 + \dfrac{\theta}{\alpha}\right]},$$

then we easily will be convinced that

$$\Delta\big(\psi_\alpha(\theta)\big) = 0, \tag{2}$$

that is the operator $\Delta(\square)$ cancels the Laplace transform of distributions $\Gamma_\alpha(x)$.

Let $\{ G_n(x),\ n \in \mathbf{N} = 1, 2, \ldots \}$, $x \ge 0$ — some sequence of distribution functions and corresponding Laplace transforms

$$\psi_n(\theta) = \int_0^\infty e^{-\theta x} dG_n(x)$$

belong to the class $\Phi$.

**Theorem.** *For the convergence*

$$\mathop{Sup}_{x} \left| G_n(x) - \Gamma_\alpha(x) \right| \to 0, \tag{3}$$

*as* $n \to \infty$, *it is necessary and sufficient that*

$$\mathop{Sup}_{\theta \le \Theta} \left| \Delta\left(\psi_n(\theta)\right) \right| \to 0, \tag{4}$$

*at any* $\Theta > 0$.

***Proof.*** Reasoning on necessity part of the condition (4) is based on properties of Laplace transform. Really, from (1) and (2) follows that

$$\begin{aligned} \left| \Delta\left(\psi_n(\theta)\right) \right| &= \left| \Delta\left(\psi_n(\theta)\right) - \Delta\left(\psi_\alpha(\theta)\right) \right| \\ &\le \left| \psi_n'(\theta) - \psi_\alpha'(\theta) \right| + \frac{1}{\alpha} \left| \psi_n^2(\theta) - \psi_\alpha^2(\theta) \right|. \end{aligned} \tag{5}$$

It is known that the distribution functions and Laplace transforms are finite functions. Hence, after differentiation and integration by parts, we have

$$\begin{aligned} \left| \psi_n'(\theta) - \psi_\alpha'(\theta) \right| &= \left| \int_0^\infty x e^{-\theta x} d\left(G_n(x) - \Gamma_\alpha(x)\right) \right| \\ &= \left| \int_0^\infty (1 - \theta x) e^{-\theta x} \left[ G_n(x) - \Gamma_\alpha(x) \right] dx \right| \\ &\le L_1 \cdot \mathop{Sup}_{x} \left| G_n(x) - \Gamma_\alpha(x) \right|, \end{aligned} \tag{6}$$

where $L_1 = L_1(\theta)$ is positive constant at any $\theta \le \Theta$. On the other hand, it is obvious, that

$$\left| \psi_n^2(\theta) - \psi_\alpha^2(\theta) \right| \le 2 \left| \psi_n(\theta) - \psi_\alpha(\theta) \right|. \tag{7}$$

From relations (5), (6) and (7), in the presence of a condition (3), we come to (4).

For sufficiency of the condition (4) we will consider (1) as the differential equation with the initial condition $\psi(0) = 1$. Then it is easy to be convinced that

$$\left| \psi_n(\theta) - \psi_\alpha(\theta) \right| = \psi_n(\theta) \psi_\alpha(\theta) \int_0^\theta \frac{\Delta\left(\psi_n(\tau)\right)}{\psi_n^2(\tau)} d\tau .$$

Whence we will receive, that

$$\mathop{Sup}_{\theta \le \Theta} \left| \psi_n(\theta) - \psi_\alpha(\theta) \right| \le L_2 \cdot \Theta \cdot \mathop{Sup}_{\theta \le \Theta} \left| \Delta\left(\psi_n(\theta)\right) \right| , \tag{8}$$

for any $\Theta > 0$, where $L_2 = L_2(\theta)$ is positive constant at any $\theta \le \Theta$.

The inequality (8) proves sufficiency of the condition (4) for convergence (3).

## Reference

[1] A.N.Tikhomirov, *On rate of convergence in the central limit theorem for weakly depending variables*, Probability Theory and its Applications, **25** (1980), **4**, pp. 800–818.

[2] Sh.K.Formanov, *Stein–Tikhomirov method and the non-classical Central Limit Theorem*, Mathematical notes, **71** (2002), **4**, pp. 604–610.