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ON THE MODELING OF THE AMERICAN OPTION PRICING

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1. We consider the financial (B,S)-market consisting only of two assets: a bank account (bonds) $B=(B_n)$ and a stocks $S=(S_n)$, where n changes from zero to N, $n=0,1,\dots,N$. According to the Cox-Ross-Rubinstein discrete model, the time-dependent change (evolution) of the values B_n and S_n is defined by the following recurrent equalities

$$B_n=(1+r)B_{n-1}, \tag{1}$$

$$S_B=(1+\rho_n)S_{n-1}, \tag{2}$$

In equalities (1), (2) it is assumed that $r>0$ is an interest rate, and ρ_n is a sequence of independent, identically distributed random variables taking only two values *a* and *b*, $-1<a<r<b$, [2], [4].

Let us now assume that there is some investor who has the initial capital $X_0=x>0$ and wants to increase this capital in the future using the capabilities of the (B,S)-market. In that case, we have the so-called investment problem. Let at the moment *n* the price of one bond is B_n and price of one stock is S_n and suppose that investor holds β_n amount of bonds and γ_n amount of stocks. Then the investor's capital can be written in the form

$$X_n^\pi=\beta_n B_n+\gamma_n S_n, \tag{3}$$

where $\pi=\pi_n=(\beta_n,\gamma_n)$ is investor's portfolio or strategy at the moment *n*.

The strategy $\pi=\pi_n=(\beta_n,\gamma_n)$ is called a American (x,f,n)-hedge if $X_0^\pi=X_0=x$, $X_n^\pi\ge f_n$, $n=0,1,\dots,N$, where $f=f_n=f_n(S_n)$ is some American option payoff function.

If we have an equality $X_n^\pi=f_n$, $n=0,1,\dots,N$, then π is called a minimal hedge. The value

$$P_n(S_n)=C_n^A=\min\{x>0:\ \Pi(x,f,n)\neq\varnothing\}, \tag{4}$$

where $\Pi(x,f,n)$ is the set of all American (x,f,n)-henges is called the fair price (or rational price) of the American option.

2. Suppose that the American option payoff function has the following form

$$f=f_n=f_n(S_n)=\beta^n\cdot(S_n-K)^+,\quad 0\le\beta\le1, \tag{5}$$

where $K>0$ is agreed price.

Lemma 1. *At each time moment n,* $n=0,1,\dots,N-1$, *a minimal strategy* $\pi_{n+1}^*=(\beta_{n+1}^*,\gamma_{n+1}^*)$ *is defined by the equalities*

$$\beta_{n+1}^*=\frac{(1+b)f((1+a)S_n)-(1+a)f((1+b)S_n)}{(1+r)(b-a)B_n}, \tag{6}$$

$$\gamma_{n+1}^*=\frac{f((1+a)S_n)-f((1+b)S_n)}{(b-a)S_n}. \tag{7}$$

Lemma 2. *The capital of the minimal strategy is defined by the equality*

$$X_n^{\pi^*} = Tf_{n+1}(S_n) = \frac{1}{1+r}\left[pf((1+b)S_n) + (1-p)f((1+a)S_n)\right], \tag{8}$$

where

$$Tf(x) = \frac{1}{1+r}\left[pf((1+b)x) + (1-p)f((1+a)x)\right], \tag{9}$$

$$p = \frac{r-a}{b-a}. \tag{10}$$

Theorem 1. *A fair* (*rational*) *price* C_n^A *of the American option* (5) *satisfies the following recurrent equation*

$$C_n^A = \max\{f_n, TC_{n+1}^A\}, \quad n = 0,1,\ldots,N-1, \quad C_N = f_N. \tag{11}$$

Theorem 2. *The rational option realization moment* τ^* (*optimal stopping moment*) *is defined by equality*

$$\tau^* = \inf\{n:\ f_n(S_n) \geq TP_{n+1}(S_n)\}. \tag{12}$$

Example. Let $f = f_2(S_n) = \max(S_n - K, 0)$, $N = 2$, $n = 0,1,2$, $B_0 = 20$, $r = \frac{1}{5}$, $S_0 = 100$, $a = -\frac{2}{5}$, $b = \frac{3}{5}$, $K = 100$, $\beta = 1$, We have:

$$S_{2,0} = 36, \quad S_{2,1} = 96, \quad S_{2,2} = 256,$$

$$f_{2,0} = 0, \quad f_{2,1} = 0, \quad f_{2,2} = 156, \quad C_{1,0}^A = 0, \quad C_{1,1}^A = 78, \quad C_2^A = 39,$$

case 1. if $S_0 \to S_{1,1} = 160$, then

$$\pi_1^* = (\beta_1^*, \gamma_1^*) = \left(-\frac{39}{20}, \frac{39}{50}\right), \quad \pi_2^* = (\beta_2^*, \gamma_2^*) = \left(-\frac{13}{4}, \frac{39}{40}\right), \quad \tau^* = 2,$$

case 2. if $S_0 \to S_{1,0} = 60$, then

$$\pi_1^* = (\beta_1^*, \gamma_1^*) = \left(-\frac{39}{20}, \frac{39}{50}\right), \quad \pi_2^* = (\beta_2^*, \gamma_2^*) = (0,0), \quad \tau^* = 1.$$

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