LAPLACE TRANSFORM OF DISTRIBUTION TO SYSTEM'S FIST OVERHAUL MOMENT

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In the paper, the obvious form of the Laplace transform of distribution of system's fist overhaul moment are found by the probability-statistical method.

Finding the distribution of the fist overhaul moment of a system is nothing but finding the distribution of the fist intersection moment of zero level by semi-Markov walk process with a negative drift, positive jumps and delays. In theory of random processes, many papers [1-5] and etc. were devoted to finding of this distribution. But a great majority of these results are less adequate to the real process or the obtained formulae are of bulky form.

In the present paper, it is assumed that no-failure operation time of the system and the size of the resource after running repair are Erlang-distributed random variables of third and second order,respectively.

Let on the probability space $(\Omega, F, P(\cdot))$ a sequence of random variables $\{\xi_k, \eta_k, \zeta_k\}_{k\geq 1}$, where ξ_k, ζ_k, η_k are positive random variables independent between them selves be given.

Let's construct from the given random variables a semi-Markov walk delay process describing the resource's level to the first overhaul time.

$$X\ (t)=\begin{cases} z-t+\sum_{i=1}^{k-1}(\zeta_i+\eta_i), & if \ \ \sum_{i=1}^{k-1}(\xi_i+\eta_i)\leq \mathrm{t}<\sum_{i=1}^{k-1}(\xi_i+\eta_i)+\xi_k, \\ z-\sum_{i=1}^{k-1}(\xi_i+\eta_i), & if \ \ \sum_{i=1}^{k-1}(\xi_i+\eta_i)+\xi_k\leq \mathrm{t}<\sum_{i=1}^{k}(\xi_i+\eta_i). \end{cases}$$

The random variable η_i means duration of the $i-$th running repair, ξ_i means the value of the used resource to the $i-$th running repair, ς_i is the gained resource after the $i-$th running repair, z is the initial value of the system's resource.

Introduce the random variable

$$\tau_0=\inf\{t : X(t)\leq 0\}$$

the first intersection moment of the zero level or the moment of the first overhaul.

Our goal is to find the Laplace transform of distribution of the random variable τ_0 and its first and second moments.

$$L(\theta|z)=E\left(e^{-\theta\tau_0}\,|X(0)=z\right),\theta>0\ .$$

$$\varphi(\theta)=E\,e^{-\theta\eta_1}.$$

COMPOSITION OF AN INTEGRAL EQUATION FOR $L(\theta|z)$.

Write a stochastic equation for the random variable τ_0.

$$\tau_0=\begin{cases} z, & if \ \ z-\xi_1<0, \\ \xi_1+\eta_1+T, & if \ \ z-\xi_1>0, \end{cases}$$

where T и τ_0 have the same distributions.

By the total probability formula we have

$$L(\theta|z)=E\left(e^{-\theta\tau_0}\,|X(0)=z\right)=\int_{\Omega}e^{-\theta\tau_0}P(d\omega|z)=\int_{\{z-\xi_1<0\}}e^{-\theta z}P(d\omega)+\int_{\{z-\xi_1>0\}}e^{-\theta(\xi_1+\eta_1+T)}P(d\omega)$$

Make the following change of variables $\xi_1 = s,\ \eta_1 = h,\ \zeta_1 = y,\ T = x.$

There with the last equation will take the form

$$L(\theta|z) = e^{-\theta z} P\{\xi_1 > z\} + \int_{s=0}^{z}\int_{h=0}^{\infty}\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-\theta s} e^{-\theta h} e^{-\theta T} P\{\xi_1 \in ds, \eta_1 \in dh, T \in dx, \varsigma_1 \in dy\} =$$

$$= e^{-\theta z} P\{\xi_1 > z\} + \varphi(\theta)\int_{s=0}^{z} e^{-\theta s}\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-\theta x} P\{T \in dx | \xi_1 = s, \zeta_1 = y\} P\{\xi_1 \in ds\} P\{\zeta_1 \in dy\} =$$

$$= e^{-\theta z} P\{\xi_1 > z\} + \varphi(\theta)\int_{s=0}^{z} e^{-\theta s}\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-\theta x} P\{T \in dx | X(0) = z - s + y\} P\{\xi_1 \in ds\} P\{\zeta_1 \in dy\}.$$

So, we get the integral equation

$$L(\theta|z) = e^{-\theta z} P\{\xi_1 > z\} + \varphi(\theta)\int_{s=0}^{z} e^{-\theta s}\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-\theta x} P\{T \in dx | X(0) = z - s + y\} P\{\xi_1 \in ds\} P\{\zeta_1 \in dy\}.$$

In the second term, having made change of variables $\alpha = z - s + y$, we get,

$$L(\theta|z) = e^{-\theta z} P\{\xi_1 > z\} - \varphi(\theta) e^{-\theta z}\int_{y=0}^{\infty} e^{-\theta y}\int_{\alpha=y}^{y+z} e^{\theta\alpha} L(\theta|\alpha)\, d_\alpha P\{\xi_1 < y + z - \alpha\} dP\{\zeta_1 < y\}.$$

If the distributions ξ_1 and ζ_1 are absolutely continuous distributions, the equation written above will take the form.

$$L(\theta|z) = e^{-\theta z} P\{\xi_1 > z\} + \varphi(\theta) e^{-\theta z}\int_{y=0}^{\infty} e^{-\theta y}\int_{\alpha=y}^{y+z} e^{\theta\alpha} L(\theta|\alpha) P_{\xi_1}(y + z - \alpha) P_{\zeta_1}(y)\, d\alpha\, dy. \quad (1)$$

We'll solve this equation in a special case for obtaining the obvious form of the solution of the last equation.

Let

$$P\{\xi_1 < t\} = 1 - (1 + \mu t) e^{-\mu t}\varepsilon(t),\ \mu > 0,$$
$$P\{\varsigma_1 < t\} = \left[1 - e^{-\lambda t}\right]\varepsilon(t),\ \lambda > 0,$$

where

$$\varepsilon(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Then, density of distribution of random variables will be

$$P_{\xi_1}(t) = \mu^2 t e^{-\mu t}\varepsilon(t),\ \mu > 0,$$
$$P_{\varsigma_1}(t) = \lambda e^{-\lambda t}\varepsilon(t),\ \lambda > 0, \text{ respectively.}$$

Under such suppositions, (1) will take the form

$$L(\theta|z) = e^{-(\mu+\theta)z}(1 + \mu z) + \lambda\mu^2\varphi(\theta) e^{-(\mu+\theta)z}\int_{y=0}^{\infty} e^{-(\mu+\theta)y}\int_{\alpha=y}^{y+z} e^{(\mu+\theta)\alpha}(y + z - \alpha) L(\theta|\alpha)\, d\alpha\, dy. \quad (2)$$

Multiply the both hand sides by $e^{(\mu+\theta)z}$.

$$e^{(\mu+\theta)z} L(\theta|z) = (1 + \mu z) + \lambda\mu^2\varphi(\theta)\int_{y=0}^{\infty} e^{-(\lambda+\mu+\theta)y}\int_{\alpha=y}^{y+z} e^{(\mu+\theta)\alpha}(y + z - \alpha) L(\theta|\alpha)\, d\alpha\, dy. \quad (3)$$

Differentiate the both hand sides of (3) with respect to z

$$(\mu + \theta) e^{(\mu+\theta)z} L(\theta|z) + e^{(\mu+\theta)z} L_z'(\theta|z) = \mu + \lambda\mu^2\varphi(\theta)\int_{y=0}^{\infty} e^{-(\lambda+\mu+\theta)y}\int_{\alpha=y}^{y+z} e^{(\mu+\theta)\alpha} L(\theta|\alpha)\, d\alpha\, dy$$

or

$$\left[(\mu+\theta)L(\theta|z)+L_z{}'(\theta|z)\right]e^{(\mu+\theta)z}=\mu+\lambda\mu^2\varphi(\theta)\int_{y=0}^{\infty}e^{-(\lambda+\mu+\theta)y}\int_{\alpha=y}^{y+z}e^{(\mu+\theta)\alpha}L(\theta|\alpha)d\alpha dy.$$

Again differentiating twice with respect to z, we'll get an ordinary differential equation of third order.

$$\left[L_z'''(\theta|z)+2(\mu+\theta)L_z''(\theta|z)+(\mu+\theta)^2L'(\theta|z)\right]e^{-\lambda z}-$$

$$-\lambda\left[L_z{}''(\theta|z)+2(\mu+\theta)L_z{}'(\theta|z)+(\mu+\theta)^2L(\theta|z)\right]e^{-\lambda z}=-\lambda\mu^2\varphi(\theta)e^{-\lambda z}L(\theta|z)$$

or

$$L_z'''(\theta|z)-[\lambda-2(\mu+\theta)]L_z''(\theta|z)-$$
$$-[2\lambda-(\mu+\theta)](\mu+\theta)L_z'(\theta|z)-\left[\lambda\mu^2[1-\varphi(\theta)]-\lambda\theta(2\mu+\theta)\right]L(\theta|z)=0. \tag{4}$$

Then, the solution and characteristic equation (4) will be

$$L(\theta|z)=c_1(\theta)e^{k_1(\theta)z}+c_2(\theta)e^{k_2(\theta)z}+c_3(\theta)e^{k_3(\theta)z}, \tag{5}$$

where $k_i(\theta),\ i=1,2,3,$ are the roots of the characteristic equation

$$k^3(\theta)-[\lambda-2(\mu+\theta)]k^2(\theta)-[2\lambda-(\mu+\theta)](\mu+\theta)k(\theta)-\lambda\mu^2[1-\varphi(\theta)]-\lambda\theta(2\mu+\theta)=0.$$

From (2) we find the boundary conditions (4).

$$L(\theta|0)=1, \tag{6}$$

$$L_z'(\theta|0)=0, \tag{7}$$

$$L_z''(\theta|0)=-(\mu+\theta)^2+\lambda\mu^2\varphi(\theta)\int_{u=0}^{\infty}e^{-\lambda u}L(\theta|u)dy. \tag{8}$$

Using (5) from (6), (7), (8) we get

$$\begin{cases} c_1(\theta)+c_2(\theta)+c_3(\theta)=1 \\ k_1(\theta)c_1(\theta)+k_2(\theta)c_2(\theta)+k_3(\theta)c_3(\theta)=0 \\ k_1^2(\theta)c_1(\theta)+k_2^2(\theta)c_2(\theta)+k_3^2(\theta)c_3(\theta)= \\ =-(\mu+\theta)^2+\lambda\mu^2\varphi(\theta)\int_{u=0}^{\infty}e^{-\lambda u}\left[c_1(\theta)e^{k_1(\theta)u}+c_2(\theta)e^{k_2(\theta)u}+c_3(\theta)e^{k_3(\theta)u}\right]du \end{cases} \tag{9}$$

Using the Viet theorem, we see that system (9) is equivalent to system (10).

$$\left.\begin{array}{l} c_1(\theta)+c_2(\theta)+c_3(\theta)=1 \\ k_1(\theta)c_1(\theta)+k_2(\theta)c_2(\theta)+k_3(\theta)c_3(\theta)=0 \end{array}\right\}$$

Let $c_3(\theta)=0.$ Then we'll define $c_1(\theta)$ and $c_2(\theta)$ from the following system:

$$\left.\begin{array}{l} c_1(\theta)+c_2(\theta)=1 \\ k_1(\theta)c_1(\theta)+k_2(\theta)c_2(\theta)=0 \end{array}\right\} \tag{10}$$

Whence we find

$$c_1(\theta)=-\frac{k_2(\theta)}{k_1(\theta)-k_2(\theta)},$$

$$c_2(\theta)=\frac{k_1(\theta)}{k_1(\theta)-k_2(\theta)}.$$

So, (5) will take the form

$$L(\theta|z)=-\frac{k_2(\theta)}{k_1(\theta)-k_2(\theta)}e^{k_1(\theta)z}+\frac{k_1(\theta)}{k_1(\theta)-k_2(\theta)}e^{k_2(\theta)z} \tag{11}$$

This is a Laplace transform of conditional distribution of the random variable τ_0. Now, find a Laplace transform of unconditional distribution of the random variable τ_0

$$L(\theta)=\int_{z=0}^{\infty} L(\theta|z)dP\{X(0)=z\}=\int_{z=0}^{\infty} L(\theta|z)d\left(1-e^{-\lambda z}\right)=\frac{\lambda\{\lambda-[k_1(\theta)+k_2(\theta)]\}}{\lambda^2-\lambda[k_1(\theta)+k_2(\theta)]+k_1(\theta)k_2(\theta)} \quad (12)$$

So, we found a Laplace transform of conditional (11) and unconditional (12) distribution of the random variable τ_0.

In the paper, we find Laplace transform of distribution to the first moment of system's overhaul.

References

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