ON A PARAMETRICAL IDENTIFICATION PROBLEM FOR NON-LINEAR EQUATIONS

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A problem of the identification of the coefficients of an object's mathematical model is investigated in the work. The coefficients depend on the phase state of the object described by a system of non-linear ordinary differential equations.

To solve the problem numerically, we propose to separate the entire set of possible phase states of the object into finite number of sub-sets. In each of these sub-sets the identifiable coefficients are sought on any parametrically given class of functions of phase. This class of functions is defined with the use of basis functions. In this case the initial problem is reduced to the identification of constant parameters participating in the representations of the coefficients.

Necessary optimality conditions of the values of the parameters defining the unknown coefficients are given in the work. The obtained formulas for the gradient of the functional allow to use iterative first order optimization methods for obtaining the values of the parameters.

Let the investigated dynamical object be generally described by a non-linear system of differential equations of the n[th] order:

$$\dot{x}(t) = f\left(x(t), K\left(x(t)\right)\right), \quad t \in \left(0, T\right], \tag{1}$$

where $x(t) \in R^n$ is a phase state vector; $K\left(x(t)\right) \in R^r$ is identifiable nearly everywhere continuously-differentiable vector-function designating the coefficients of the mathematical model; the known vector-function $f(.,.)$ is continuously-differentiable on all its arguments.

Suppose that in the aim of identifying the coefficients of the mathematical model of process (1), N independent observations have been carried out over the dynamics of the process at different initial conditions:

$$x^i(0) = x_0^i, \quad i = 1,2,..., N . \tag{2}$$

The results of observations may also be some components or the whole state vector at different moments of time

$$x\left(t_{ij}; x_0^i\right) = x^{ij}, \quad t_{ij} \in \left(0, T\right], \quad j = 1,2,..., M_i, \quad i = 1,2,..., N , \tag{3}$$

particularly, at the final moment of time T

$$x\left(T; x_0^i\right) = x_T^i, \quad i = 1,2,..., N , \tag{4}$$

where M_i is the number of moments of time at which observations have been carried out over the state of the object with initial condition x_0^i at i th experiment. There may also be observations over the state of the object at different initial conditions at some time intervals:

$$x\left(t; x_0^i\right) = y^{ij}(t), \quad t \in \left[\tau_{ij-1}, \tau_{ij}\right] \subset \left[0, T\right], \quad \tau_{ij-1} < \tau_{ij}, \quad j = 1,2,..., M_i, \quad i = 1,2,..., N , \tag{5}$$

where M_i is the number of time intervals at initial condition x_0^i, at which observations have been carried out over the object. Observations may also be of mixed type, i.e. both pointwise (3) or (4) and interval (5).

The considered problem consists in determining (identifying) the unknown coefficients $K(x)$ of system (1) in the presence of the results of observations (2), (3), (4) or (5).

The quality of identification is estimated with the use of a mean-square performance criterion, at that for each type of observations (3)-(5) the specific form of the criterion is different. For the sake of concreteness we assume that final observations (4) have been carried out. Then the performance criterion is as follows:

$$J(K(x)) = \frac{1}{N} \cdot \sum_{i=1}^{N} I^i\left(x\left(T; x_0^i, K(x)\right)\right) + \varepsilon \cdot \left\|K(x) - \hat{K}(x)\right\|_{L_2^r}^2 \to \min_{K(x)}, \tag{6}$$

$$I^i\left(x\left(T; x_0^i, K(x)\right)\right) = \left\|x\left(T; x_0^i, K(x)\right) - x_T^i\right\|_{R^n}^2, \tag{7}$$

where $x(t) = x(t; x_0, K(x))$ is the solution to problem (1) in the presence of some initial condition x_0 and coefficients defined by the vector-function $K(x) \in R^r$; $\varepsilon > 0$, $\hat{K}(x)$ are regularization parameters.

In order to restore the unknown coefficients of the system of differential equations (1), we propose the following approach. The entire set of possible phase states is separated into finite number of sub-sets. In each of these sub-sets the coefficients are sought on a parametrically given class of functions of state defined with the use of basis functions

Denote by $X \subseteq R^n$ a set of all possible phase states of the object $x(t)$ in the presence of all admissible values of the initial conditions and the values of the coefficients $K(x)$. Let X be covered by the given finite number L of simply connected sub-sets (zones) $X^k \subset X$ so that

$$X \subseteq \bigcup_{\nu=1}^{L} X^\nu, \ \text{int}\, X^i \bigcap \text{int}\, X^j = \varnothing, \ \ i, \ j = 1,2,...,L, \ \ i \neq j. \tag{8}$$

The zones of phase space

$$\begin{aligned} &X^\nu = \left\{x \in R^n : g^{\nu-1}(x) > 0, \quad g^\nu(x) \le 0\right\}, \quad \nu = 2,3,...,L-1, \\ &X^1 = \left\{x \in R^n : g^1(x) \le 0\right\}, \quad X^L = \left\{x \in R^n : g^{L-1}(x) > 0\right\}, \end{aligned} \tag{9}$$

are defined by their boundaries with the use of the given continuously-differentiable functions $g(x) = \left(g^1(x), g^2(x), ..., g^{L-1}(x)\right)$.

We suppose that the vector-functions $f(.,.)$ and $g(.)$ satisfy the following conditions:

$$\|f(x,K)\| \le m_1, \ \ \|\nabla_x f(x,K)\| \le m_2, \ \ \|\nabla_K f(x,K)\| \le m_3,$$

$$\left|g^\nu(x)\right| \le m_4, \ \ \left\|\nabla g^\nu(x)\right\| \le m_5. \tag{10}$$

when $t \in [0,T]$ and $x \in X$, $\nu = 1,2,...,L-1$, where $m_i, i = 1,2,...,5$, are given positive numbers.

The identifiable coefficients $K(x) = (k_1(x),...,k_r(x))$ in each of the zones X^ν, $\nu = 1,2,...,L$, are defined as follows:

$$K(x) = K^\nu(x) \in R^r, \ \ k_s^\nu(x) = \sum_{i=1}^{m} p_{si}^\nu \cdot \varphi_i(x), \ s = 1,2,...,r, \tag{11}$$

$$p_{si}^\nu = const, \ s = 1,2,...,r, \ i = 1,2,...,m, \ x \in X^\nu, \ \nu = 1,2,...,L, \ \ t \in [0,T],$$

where $\varphi_i(x)$, $i = 1,2,...,m$, are given linearly-independent basis functions; p_{si}^ν are as yet unknown constant parameters designating the identifiable functions. As a rule, the coefficients $K(x)$ in real problems must satisfy some constraints arising from technical or technological considerations. Then the parameters $p = \left(p^1, p^2, ..., p^L\right)$, $p^\nu = \left(p_{11}^\nu, ..., p_{1m}^\nu, ..., p_{rm}^\nu\right)$, $\nu = 1,2,...,L$, must also satisfy the definite corresponding constraints. Denote by $P^\nu \subset R^{r \times m}$, $\nu = 1,2,...,L$ the sets of admissible values of the zonal parameters p^ν. These sets are supposed to be closed and bounded. $P = P^1 \times P^2 \times ... \times P^L$.

In this case the solution to differential equations system (1), defining the current state of the process $x(t)$, will depend on the initial condition x_0 and on zonal values of the vector of parameters p, i.e. $x(t) = x(t; x_0, p)$.

Taking into account representation (11) the identification criterion will be as follows:

$$J(p)=\frac{1}{N}\cdot\sum_{i=1}^{N} I^i\left(x\left(T;x_0^i,p\right)\right)+\varepsilon\cdot\left\|p-\hat{p}\right\|^2_{R^{L\times r\times m}}\to\min_{p\in P},$$
$$I^i\left(x\left(T;x_0^i,p\right)\right)=\left\|x\left(T;x_0^i,p\right)-x_T^i\right\|^2_{R^n}, \quad (12)$$

where $x(t)=x(t;x_0,p)$ is the solution to Cauchy problem (1) in the presence of the given admissible vector of parameters p and initial condition x_0 taking into account (11); $\hat{p}$ is a regularization parameter corresponding to the function $\hat{K}(x)$ from (6).

Problem (1), (2), (4), (12) can be related to a parametrical optimal control problem. At the same time this problem, owing to the fact that the finite-dimensional vector of parameters p is optimized, can be related to finite-dimensional optimization problems. In order to solve these problems, we can make use of efficient numerical methods, particularly, of first order and the finished standard software packages [2, 3]. Thereto, we need to obtain formulas for the components of the gradient of target functional (12) with respect to the components of the vector p – $\nabla_p J(p)$. These formulas will also allow us to formulate necessary first order optimality conditions with respect to the vector of parameters p.

Let p be an admissible value of the parameter. Suppose that the input data and functions participating in the statement of the considered problem are such that for arbitrary admissible values of the vector p the trajectory of the system when hitting the boundary surface never slides over it, i.e. there always holds true the condition

$$\left|\left(g_x^{\nu *}\left(x\left(\bar{t}_\nu\right)\right), f\left(x\left(\bar{t}_\nu\right), K^\nu(x)\right)\right)\right|\geq\delta>0,\quad \nu=1,2,...,L-1. \quad (13)$$

Here «*» is transposition sign; $\bar{t}_\nu\in[0,T]$ are moments of time when the trajectory hits the boundary surface; at that the point and moment of intersection is stable to small perturbations of the parameters of the problem. This condition is not of principal value, but the case when it does not hold true necessitates carrying on additional computations for the sections of the trajectory which are on the boundary surface.

The following remark is of important value. It is evident that the experiments and the results of observations (4) are independent of each other. The same is true for the items of the functional (12). This means that the following formula takes place for the gradient of the functional:

$$\nabla_p J(p)=\frac{1}{N}\cdot\sum_{i=1}^{N}\nabla_p I^i\left(x\left(T;x_0^i,p\right)\right)+2\cdot\varepsilon\cdot\left(p-\hat{p}\right).$$

That is why in order to obtain formulas for $\nabla J(p)$ we need to obtain formulas for the gradient with respect to individual items $\nabla_p I^i\left(x\left(T;x_0^i,p\right)\right)$, $i=1,2,...,N$. To this end we use the formula of the increment of target functional (12) that is obtained at the expense of the increment of the values of the parameters p.

In the general case for arbitrary number of zones of the phase space, i.e. at $L>2$ we obtain the following formulas for the components of the gradient of the target functional

$$\frac{d}{dp_{kj}^l}J(p)=\sum_{i=1}^{N}\frac{d}{dp_{kj}^l}I^i\left(x\left(T;x_0^i,p\right)\right)+2\cdot\varepsilon\cdot\left(p_{kj}^l-\hat{p}_{kj}^l\right),$$
$$\frac{d}{dp_{kj}^l}I^i\left(x\left(T;x_0^i,p\right)\right)=\int\limits_{\Pi_l\left(x_0^i;p\right)}\left[\psi^*\left(t;x_0^i,p\right)\cdot\frac{\partial f\left(x\left(t;x_0^i,p\right),K^l\right)}{\partial K^l}\cdot\frac{\partial K^l}{\partial p_{kj}^l}\right]dt, \quad (15)$$
$$\frac{\partial K^l}{\partial p_{kj}^l}=\varphi_j(x),\ k=1,2,...,r,\ j=1,2,...,m,\ l=1,2,...,L,$$

where $\Pi_l\left(x_0^i; p\right)$, $l = 1,2,...,L$, $i = 1,2,...,N$, is the time interval during which the trajectory with initial condition x_0^i and the value of the parameter p was in zone X^l; $\psi\left(t; x_0^i, p\right)$ is the solution to the following conjugate system

$$\dot{\psi}^*\left(t; x_0^i, p\right) = -\psi^*\left(t; x_0^i, p\right) \cdot \frac{df\left(x(t), K^\nu\right)}{dx}, \; t \in \Pi_l\left(x_0^i; p\right), \quad (16)$$

$$\psi(T) = \frac{\partial I\left(x\left(T; x_0, p\right)\right)}{\partial x}. \quad (17)$$

satisfying the following jump condition at the moment of time when the trajectory of system (1) hits the boundary surface:

$$\psi\left(\bar{t}_\nu - 0\right) = \psi\left(\bar{t}_\nu + 0\right) - \frac{\partial g\left(x\left(\bar{t}_\nu\right)\right)}{\partial x} \cdot \gamma, \; \gamma = \frac{\psi^*\left(\bar{t}_\nu + 0\right) \cdot \left[f\left(x\left(\bar{t}_\nu\right), K^\nu\right) - f\left(x\left(\bar{t}_\nu\right), K^{\nu+1}\right)\right]}{\frac{\partial g^*\left(x\left(\bar{t}_\nu\right)\right)}{\partial x} \cdot f\left(x\left(\bar{t}_\nu\right), K^\nu\right)}, \quad (18)$$

$$\nu = 1,2,...,L-1.$$

Remark. If in the presence of some admissible values of the vector p the trajectory of the system stays in one of the zones at $t \in [0, T]$, then the gradient of the functional and conjugate system will be defined by the known formulas for systems without switchings [5, 6], i.e. the integration in formula (15) will be carried out on the interval $[0, T]$, and $\psi(t)$ will satisfy conjugate system (16), (17) everywhere on $[0, T]$ without jump condition (18).

The following theorem holds true.

Theorem (necessary optimality condition). For the optimality of the vector $\tilde{p}$ in problem (1), (2), (4), (8)-(12), it is necessary that the following relation be satisfied

$$\nabla_p J(p) \cdot \left(\tilde{p} - p\right) = \sum_{i=1}^{N} \nabla_p I^i\left(x\left(T; x_0^i, p\right)\right) \cdot \left(\tilde{p} - p\right) \geq 0, \; \forall p \in U\left(\tilde{p}, \alpha\right),$$

where $\nabla_p J(p)$ is defined by formula (15); $U\left(\tilde{p}, \alpha\right)$ is α-neighborhood of the point $\tilde{p}$.

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