MARTINGALE MEASURES FOR THE GEOMETRICAL GAUSSIAN MARTINGALE

Omar Glonti[1], Zaza Khechinashvili[2]

Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia
[1]*omglo@yahoo.com*, [2]*khechinashvili@mail.ru*

On the filtered probability space $(\Omega, F, (F)_{0\le n\le N}, P)$ consider the stochastic process of discrete time

$$S_n = S_0 \exp\{M_n\}, \;\; n = 1,...,N,$$

where $S_0 > 0$ is deterministic, $(M_n, F_n)_{0\le n\le N}$, $M_0 = 0$, is a Gaussian martingale with quadratic characteristic $\langle M\rangle_n = EM_n^2$. We describe evolution of risky asset by this scheme.

Define

$$Z_n^{\psi} = Z_{n-1}^{\psi} \frac{\exp\{\psi_n\}}{E[\exp\{\psi_n\}/F_{n-1}]}, \;\; Z_0 = 1,$$

where $(\psi_n, F_n)_{0\le n\le N}$ some stochastic sequence. It is clear, that Z_n^{ψ} is F_n measurable and if $E\exp\{\psi_n\} < \infty$, then $(Z_n, F_n)_{0\le n\le N}$ is P martingale and represents the density process.

Consider measure

$$Q_{\psi}(A) = \int_A Z_N^{\psi}(\omega) dP(\omega), \; A \in F,$$

which is equivalent to P.

If $\psi = (\psi_n, F_n)$ satisfies

$$E[\exp\{\psi_n + \Delta M_n\}/F_{n-1}] = E[\exp\{\psi_n\}/F_{n-1}], (P - a.s), \tag{1}$$

then Q_{ψ} is a martingale measure for S, i.e. Q_{ψ} is equivalent to P and S is (F_n, Q_{ψ}) martingale.

Let $\psi_n = a\Delta M_n^2 + b\Delta M_n$, where a and b are some constants. In this case the condition (1) has form

$$E\exp\{a\Delta M_n^2 + (b+1)\Delta M\} = E\exp\{a\Delta M_n^2 + b\Delta M\}$$

and $b = -\frac{1}{2}$.

So, the condition (3) is fulfilled if $b = -\frac{1}{2}$ and for any constant a the class of martingale measures for S is defined by density process

$$Z_n = Z_{n-1} \frac{\exp\{a(\Delta M_n)^2 - \frac{\Delta M_n}{2}\}}{E\exp\{a(\Delta M_n)^2 - \frac{\Delta M_n}{2}\}} = \prod_{k=1}^{n} \frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}, \tag{2}$$

Let's consider the class of martingale measures Q^a $(a \in R)$ with density (2), i.e.

$$\frac{dQ_a}{dP} = Z_N = \prod_{k=1}^{N} \frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}. \qquad (3)$$

Our aim is to find the constant a^* and corresponding probability measure Q_a^* which minimizes the relative entropy.

Recall, that the relative entropy of probability measure Q with respect to probability measure P is defined as

$$I(Q,P) = \begin{cases} E_P[\frac{dQ}{dP}\ln\frac{dQ}{dP}], & if Q << P, \\ \infty, otherwise. \end{cases}$$

So we have to find constant a^* and corresponding measure Q_a^* with density (3) for which

$$I(Q_a^*, P) \to \min.$$

The following theorem is true

Theorem. Let $S_n = S_0 \exp\{M_n\}, n = 1,..., N, S_0 > 0,$ where $(M_n, F_n)_{0\le n\le N}, M_0 = 0$ is the gaussian martingale with quadratic charasteristic $\langle M\rangle_n = EM_n^2$. In class of martingale measures with densities defined by (3) the minimal relative entropy martingale measure Q_a^* has the density

$$\frac{dQ_a^*}{dP} = \prod_{k=1}^{N} \frac{\exp\{\frac{1-\sqrt{3}}{4}(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E\exp\{\frac{1-\sqrt{3}}{4}(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}.$$

References

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