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ON FUNCTIONAL LIMIT THEOREM FOR STOCHASTIC BRANCHING PROCESSES WITH IMMIGRATION

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Let $\{\xi_{kj},\varepsilon_k,k,j\in N\}$ be independent, nonnegative integer-valued random variables such that $\{\xi_{kj},\ k,j\in N\}$ and $\{\varepsilon_k,k\in N\}$ are indentically distributed. Define the sequence of random variables $\{X_k,k\geq 0\}$ by the following recurrence relations

$$X_0=0,\ \ X_k=\sum_{j=1}^{X_{k-1}}\xi_{kj}+\varepsilon_k,k=1,2...$$

The sequence $\{X_k,k\geq 1\}$ is called a stochastic branching process with immigration. We can interpret X_k as the size of the *k*-th generation of a population, where $\xi_{k,j}$ is the number of offspring of the j th individual in the (k-1)-th generation and ε_k is the number of immigrants contributing to the k th generation. We assume $E\left((\xi_{1,1})^2+(\varepsilon_1)^2\right)<\infty$ and denote $m=E\xi_{1,1},\ \sigma^2=\operatorname{var}\xi_{1,1},\ \lambda=E\varepsilon_1,\ \beta^2=\operatorname{var}\varepsilon_1$

The cases $m<1$, m=1 and $m>1$ are referred to respectively as subcritical, critical and supercritical. In the critical case $m=1$, Wei and Winnicki [1] considered the random step function $X_{[nt]},\ t\geq 0$ as random elements in the Skorokhod Space $D[0,\infty)$, where [*a*] denotes the lower interger part *a* , and proved the weak convergence $n^{-1}X_{[nt]}\to X(t)$ as $n\to\infty$ in the $D[0,\infty)$, where *X* is unique solution to the stochastic differential equation

$$dX(t)=\lambda\, dt+\sigma\sqrt{X(t)}dW(t)$$

with initial condition $X(0)=0$, where *W* is a standard Wiener process.

For each $n\in N$ let $\{\xi_{kj}^{(n)},\varepsilon_k^{(n)},k,j\in N\}$ be independent, nonnegative integer-valued random variables such that $\{\xi_{kj}^{(n)},\ k,j\in N\}$ and $\{\varepsilon_{kj}^{(n)},\ k,j\in N\}$ are identically distributed. We consider a sequence of branching processes with immigration $\{X_k^{(n)},\ k\in N\}$, $n\in N$ given by the recurrence relations

$$X_0^{(n)}=0,\ \ X_k^{(n)}=\sum_{j=1}^{X_{k-1}^{(n)}}\xi_{kj}^{(n)}+\varepsilon_k^{(n)},\ k,n\in N$$

Assume that $m_n=E\xi_{1,1}^{(n)},\ \lambda_n=E\varepsilon_1^{(n)},\ \sigma_n^2=\operatorname{var}\xi_{1,1}^{(n)},\ \beta_n^2=\operatorname{var}\varepsilon_1^{(n)}$ are finite for all $n\in N$. Now introduce the random step functions $X_n(t)=X_{[nt]},\ t\geq 0,\ n\in N$. The convergence of finite – dimentional distributions of a sequence of branching processes with immigration has been investigated by Kawazu and Watanabe [2] and Aliev [3]. Under the assumptions that

1) $m_n=1+\alpha n^{-1}+o\left(n^{-1}\right)$ as $n\to\infty$ for some $\alpha\in R$

2) $\sigma_n^2\to 0$ as $n\to\infty$

3) $\lambda_n \to \lambda \ge 0$ and $b_n^2 \to b^2 \ge 0$ as $n \to \infty$ in the paper [4] proved weakly in the Skorokhod space $D[0,\infty)$

$n^{-1}X_n(t) \to m(t)$ as $n \to \infty$ where $m(t) = \lambda \int_0^t e^{\alpha s} ds,\ t \ge 0$ and also obtained that sequence

$n^{-\frac{1}{2}}\left(X_n(t) - EX_n(t)\right)$ has a limit process $X(t)$ as $n \to \infty$. Process $X(t)$ is the unique solution of the stochastic differential equation

$dX(t) = \alpha X(t)dt + \sqrt{\rho(t)}dW(t),\ X(0) = 0$, where $\rho(t) = b^2 + \sigma_0^2 \lambda \int_0^t e^{\alpha s} ds.$

We also investigate the sequence $X_n(t)$ and prove next result.

Theorem. Suppose that

1) $m_n = 1 + \alpha d_n^{-1}$ for some $\alpha \in R$, where d_n is the sequence of positive members such that $nd_n^{-1} \to \beta < \infty$,

2) $n\lambda_n \to \lambda \ge 0,\ n\sigma_n^2 \to \sigma^2 \ge 0,\ b_n^2 \to b^2 \ge 0$ as $n \to \infty$

3) $E\left(\varepsilon_1^{(n)} - \lambda_n\right)^2 I\left(\left|\varepsilon_1^{(n)} - \lambda_n\right| > \theta\sqrt{n}\right) \to 0$ as $n \to \infty$ for each $\theta > 0$, where $I(A)$ is indicator of A.

Then, weakly in the Skorokhod space $D[0,T]$

$n^{-\frac{1}{2}}\left(X_n(t) - EX_n(t)\right) \to Z(t)$ as $n \to \infty$, where $Z(t) = b\int_0^t e^{\beta\alpha(t-s)} dW(s)$.

References

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