

FOURIER-BASED METHOD FOR ESTIMATING SIGNAL PERTURBATIONS IN LINEARLY-CORRELATED NOISE

**Irina Gorodnitsky¹, Tahir Alizada², Sevinj Guliyeva³,
Dilara Mastaliyeva⁴, Ulkar Rzayeva⁵**

¹University of California, San Diego, USA, *igorodni@ucsd.edu*

²⁻⁵Institute of Cybernetics of ANAS, Baku, Azerbaijan

²*TahirAlizada@gmail.com*, ³*sevinc-guliyeva@rambler.ru*,

⁴*dilara.mestelieva@gmail.com*, ⁵*ulker.rzaeva@gmail.com*

Signal estimation in the presence of correlated noise is difficult and standard methods often produce estimates that are sub-optimal [1-7]. In this paper a novel method is presented for estimating the deterministic signal $x(i\Delta t)$ and the additive random noise $\varepsilon(i\Delta t)$ when the two processes are correlated. It has been shown previously [1-7] that our ability to correct for distortions in signal estimates that are caused by correlated noise can be reduced to the problem of estimating the noise variance D_ε . To elaborate on this result, we consider a noisy measurement $g(i\Delta t) = x(i\Delta t) + \varepsilon(i\Delta t)$ and we also denote the coefficients of the respective Fourier decomposition of $x(i\Delta t)$ and $g(i\Delta t)$ as $\{a_n, b_n\}$ and $\{a_n^*, b_n^*\}$:

$$a_n = \frac{2}{N} \sum_{i=1}^N x(i\Delta t) \cos n\omega(i\Delta t), \quad b_n = \frac{2}{N} \sum_{i=1}^N x(i\Delta t) \sin n\omega(i\Delta t) \quad (1)$$

$$a_n^* = \frac{2}{N} \sum_{i=1}^N g(i\Delta t) \cos n\omega(i\Delta t), \quad b_n^* = \frac{2}{N} \sum_{i=1}^N g(i\Delta t) \sin n\omega(i\Delta t),$$

where ω indicates the chosen frequency resolution of the Fourier decomposition. The DC-term exist, Eq. (1) has an additional component $a_0/2$. To streamline the discussion, we derive results for the case of linearly correlated noise which can be extended to the general case in a straightforward way.

Let's assume that an estimate $x^*(t)$ is found using any one of the standard filtering methods that successfully remove the stochastic component of noise but do not affect the distortion in the estimate that is due to the signal-noise correlation. By definition, a linear distortion due to noise in the signal estimate is related to the true signal $x(t)$ as:

$$k = \frac{x(t)}{x^*(t)}. \quad (2)$$

The constant k has been termed a distortion parameter [1, p.181; 2, p.27]. If we can find k , we can recover the true signal $x(t)$ from $x^*(t)$. Note that in our formulation, the samples $x^*(i\Delta t)$ are characterized by a subset of Fourier coefficients of $g(i\Delta t)$ given in Eq. (1) that correspond to the signal-noise correlated part of data. In other words, when noise has quasnormal distribution, we expect that a standard method used to compute $x^*(t)$ will remove those a_n^* and b_n^* coefficients that correspond to the stochastic part of the noise while the remaining coefficients from Eq. (1) will be equal to the Fourier coefficients of the estimated sequence $x^*(t)$, i.e. $a_k^* = \frac{2}{N} \sum_{i=1}^N x^*(i\Delta t) \cos k\omega(i\Delta t)$, where k is a subset of numbers n in Eq.

(1). Note that the estimate $x^*(t)$ is also related to the true signal $x(t)$ via the following equalities $a_n = ka_k^*$ and $b_n = kb_k^*$.

We propose the following method for finding k . Let's consider the additive noise, which satisfies

$$\varepsilon(i\Delta t) = g(i\Delta t) - x(i\Delta t) = g(i\Delta t) - kx^*(i\Delta t). \quad (3)$$

Squaring each side and summing across the length of the sequences yields:

$$\sum_{i=1}^N \varepsilon^2(i\Delta t) = \sum_{i=1}^N g^2(i\Delta t) - 2k \sum_{i=1}^N g(i\Delta t)x^*(i\Delta t) + k^2 \sum_{i=1}^N x^{*2}(i\Delta t) \quad (4)$$

Eq. (4) contains only two unknowns, the noise variance D_ε and the correction k . Thus, given D_ε we can find a closed form solution for k . To simplify the notation, we denote the variance of $g(i\Delta t)$ as D_g , the variance of $x^*(i\Delta t)$ as D_{x^*} , and the correlation between $g(i\Delta t)$ and $x^*(i\Delta t)$ as R_{gx^*} . Eq. (4) then becomes:

$$D_\varepsilon = D_g - 2kR_{gx^*} + k^2D_{x^*} \quad (5)$$

The discriminant of the square equation (5) is

$$D = 4R_{gx^*}^2 - 4D_{x^*}(D_g - D_\varepsilon). \quad (6)$$

and candidate values of k are given by the roots of Eq. (5)

$$k_{1,2} = \frac{R_{gx^*} \pm \sqrt{R_{gx^*}^2 - D_{x^*}(D_g - D_\varepsilon)}}{D_{x^*}}. \quad (7)$$

Eq. (7) yields two candidate values for k . The correct candidate can be selected by computing $a_n(i) = k_{1,2}a_k^*$, $b_n(i) = k_{1,2}b_k^*$, and selecting the value k_i , $i=\{1,2\}$, which produces estimates of $a_n(i)$ and $b_n(i)$ that are closest to a_k^* and b_k^* .

From Eq. (7), the limiting factor in our ability to improve on the estimate $x^*(i\Delta t)$ is the accuracy of the noise variance estimate D_ε . The only characteristic that distinguishes the signal of interest and noise in our formulation is the presence of a stochastic component in the noise but not in the signal. Thus methods for estimating D_ε are based on identifying those data samples that contain mostly noise, i.e. $x(i\Delta t) \approx 0$, based on statistical properties of the samples, and then estimating D_ε from those samples. For example, the formula in [6, pp.52,55-74] based on this approach provides:

$$D_\varepsilon = \overline{g(i\Delta t)^2}^+ + \overline{g(i\Delta t)g((i+2)\Delta t)}^+ - 2\overline{g(i\Delta t)g((i+1)\Delta t)}^+, \quad (8)$$

where $\overline{(\cdot)}^+$ indicates data samples that contain primarily noise. Once D_ε is found, k can be computed as shown above. In the following, we examine the behavior of the distortion parameter k in the two limiting conditions.

Case 1: Zero noise: In negligible noise

$$g(i\Delta t) \approx x^*(i\Delta t) \approx x(i\Delta t). \quad (9)$$

Then

$$D_g = D_{x^*} = D_x, \quad (10)$$

$$R_{gx^*} = R_{xx} = D_x. \quad (11)$$

Then, Eq. (7) becomes

$$D \approx 4D_x^2 - 4D_x(D_x - D_\varepsilon). \quad (12)$$

Since $D_\varepsilon \approx 0$ when noise is negligible, we have $D = 0$. The estimate of the distortion coefficient Eq. (7) becomes

$$k_{1,2} = \frac{R_{gx^*} \pm \sqrt{D}}{D_{x^*}} = \frac{D_x}{D_x} = 1, \quad (13)$$

as expected.

Case 2: White noise: Since white noise is removed by standard filtering techniques, we expect $x^*(i\Delta t)$ to provide an accurate signal estimate, i.e. $x^*(i\Delta t) \approx x(i\Delta t)$. In this case

$$D_{x^*} = D_x, \text{ while } D_g \neq D_x, D_\varepsilon > 0. \quad (14)$$

From [6]

$$R_{gx} = D_x, \quad (15)$$

$$D_g = D_x + D_\varepsilon. \quad (16)$$

Substituting these into Eq. (6) we get

$$D = 4D_x^2 - 4D_x(D_g - D_\varepsilon) = 0 \quad (17)$$

and Eq. (7) becomes

$$k_{1,2} = \frac{R_{gx^*} \pm \sqrt{D}}{D_{x^*}} = \frac{D_x}{D_x} = 1, \quad (18)$$

as expected.

Discussion

When noise is white or when it is negligible, it does not affect statistics of a deterministic signal of interest. This signal can be estimated by removing any existing noise using standard filtering methods. In practice, however, noise is often colored and the signal and noise are often correlated. The discriminant D can be used as an indicator of this correlation. As was shown in this paper, $D \approx 0$ is an indicator that noise is either absent or is not correlated with the signal. Moreover, if a fairly accurate estimate of the noise variance D_ε can be found, we can correct for the errors in the standard signal estimate $x^*(t)$ to find a more accurate approximation to $x(t)$.

In general, the assumption of linear correlation is idealistic. Noise is expected to scale the different Fourier modes of the signal unequally, resulting in non-linearly correlated noise. We can adapt the methodology described above to this general case by considering the individual Fourier components of data in place of the entire data sequence and computing a correction k_n for each individual data mode. (In practice, we would consider only the modes that contain energy above a preset threshold). We can obtain a derivation similar to that above using estimates of variance and correlation coefficients for each individual spectral component in place of those obtained for the entire signal above. The correction k_n for each Fourier component n would satisfy the relationships $a_n = k_n a_n^*$, $b_n = k_n b_n^*$, which we can use to correct the Fourier coefficients of $x^*(t)$ and hence obtain a more accurate estimate of $x(t)$.

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