SEMI-RECURSIVE KERNEL ESTIMATION OF THE PRODUCTION FUNCTION AND ITS CHARACTERISTICS

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The general approach for nonparametric estimation of the production function and its characteristics depending on the function's partial derivations is proposed. This approach is based on ideas of estimations of functions depending on multivariate density functionals and their derivatives [1]. The problem of obtaining of the main parts of the asymptotic mean square errors (MSE) of the estimates also resolved by using a piecewise smooth approximation. A regression model of the production function is used.

Let us consider the function

$$J(x) = H\left(\{a_i(x)\}, \{a_i^{(1j)}(x)\}, i=\overline{1,s+1},\ j=\overline{1,m}\right) = H\left(x, \{a(x)\}, \{a^{(1j)}(x)\}\right), \qquad (1)$$

where $x \in R^m$, $H(t): R^{(m+1)(s+1)} \to R$ is a given function, basic functionals and derivatives

$$a_i^{(0j)}(x) = a_i(x) = \int_R g_i(y) f(y,x)dy, \quad a_i^{(1j)}(x) = \frac{\partial a_i(x)}{\partial x_j},\ i=\overline{1,s+1},\ j=\overline{1,m}, \quad g_1,...,g_s$$

are known Borel functions, $g_{s+1} \equiv 1$, $f(x,y)$ is an unknown probability density function (p.d.f.) for the observed random vector $Z=(X,Y)\in R^{m+1}$. Denote conditional functionals

$$b_i^{(0j)}(x) = b_i(x) = a_i(x)/p(x) = a_i(x)/a_{s+1}(x) = \int_R g_i(y) f(y\,|\,x)dy,\ b_i^{(1j)}(x) = \frac{\partial b_i(x)}{\partial x_j},$$

$i=\overline{1,s}$, where $p(x)$ is the marginal probability density of the random variable *X,* and $f(y\,|\,x) = f(x,y)/p(x)$ is the conditional probability density.

According to above given the regression model of the production function

$$r(x) = \mathbf{E}(Y\,|\,X=x) = \mathbf{E}(Y\,|\,x) = \int_R yf(y\,|\,x)dy = \frac{\int_R yf(y,x)dy}{p(x)};$$

$$H(a_1,a_2) = a_1/a_2, g_1(y) = y;$$

the marginal productivity function $MP_j(x)$

$$MP_j(x) = \frac{\partial r(x)}{\partial x_j};\ H\left(a_1,a_2,a_1^{(1j)},a_2^{(1j)}\right) = \frac{a_1^{(1j)}}{a_2} - \frac{a_1 a_2^{(1j)}}{a_2^2} = b_1^{(1j)},\ g_1(y)=y,$$

the marginal rate of technical substitution of an input x_j with an input x_i

$$MRTS_{ij}(x) = \frac{MP_i(x)}{MP_j(x)};\quad H\left(a_1,a_2,a_1^{(1j)},a_2^{(1j)},a_1^{(1i)},a_2^{(1i)}\right) = \frac{b_1^{(1i)}}{b_1^{(1j)}}.$$

Take the following expression as an estimate of the functional $a(x)=a^{(0j)}(x)$ $(r=0)$ and its derivatives $a^{(1j)}(x)$ $(r=1)$ at a point *x*:

$$a_n^{(rj)}(x) = \frac{1}{n}\sum_{l=1}^n h_l^{-m-r} g(Y_l)\mathbf{K}^{(rj)}\left(\frac{x-X_l}{h_l}\right) =$$
$$= a_{n-1}^{(rj)}(x) - \frac{1}{n}\left[a_{n-1}^{(rj)}(x) - h_n^{-m-r} g(Y_n)\mathbf{K}^{(rj)}\left(\frac{x-X_l}{h_l}\right)\right]. \qquad (2)$$

Here $Z_l = (X_l, Y_l),\quad l = \overline{1,n}$, is the $(m+1)$ -dimensional random sample from p.d.f. $f(x,y)$, $(h_n) \downarrow 0$ is a sequence of positive bandwidths such that $\frac{1}{n}\sum_{i=1}^{n} h_i^{\lambda} = S_{\lambda} h_n^{\lambda} + o\left(h_n^{\lambda}\right)$, where λ is a real number, S_{λ} is some constant independent on *n*, $\mathbf{K}^{(0j)}(u/h_l) = \mathbf{K}(u/h_l) = \prod_{i=1}^{m} K(u_i/h_l)$ is a *m*-dimensional multiplicative kernel function, $\int_R K(u)du = 1,\ K(u) = K(-u)$,

$$\mathbf{K}^{(1j)}(u) = \frac{\partial \mathbf{K}(u)}{\partial u_j} = K(u_1)...K(u_{j-1})K^{(1)}(u_j)K(u_{j+1})...K(u_m),\ K^{(1)}(u_j) = \frac{dK^{(1)}(u_j)}{du_j},$$

$$g(y) = \left(g_1(y),...,g_{s+1}(y)\right).$$

Recursive estimation is particularly useful in large sample size since (2) can be easily updated with each additional observation. The recursive kernel estimate of $p(x)$ was introduced in [2] and apparently independently in [3], and has been thoroughly examined in [4].

Semi-recursive kernel type estimates of the conditional functionals $b(x) = \left(b_1(x),...,b_s(x)\right)$ at a point *x* are designed as

$$b_n(x) = \sum_{l=1}^{n} \frac{1}{h_l^m} g(Y_l)\mathbf{K}\left(\frac{x - X_l}{h_l}\right) / \sum_{l=1}^{n} \frac{1}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right) = \frac{a_n(x)}{p_n(x)} = \frac{a_n^{(0j)}(x)}{a_{(s+1)n}^{(0j)}(x)}. \quad (3)$$

The substitution estimates are often used for the estimation of ratios. The possible unboundedness of the ratio estimates at some points (see [5] for details) creates a difficulty in the obtaining their MSE. The estimates (3) are called semi-recursive because they can be updated sequentially by adding extra terms to both the numerator and denominator when new observations became available. We can rewrite (3) as

$$b_n(x) = b_{n-1}(x) + \left(g(Y_n) - b_{n-1}(x)\right)\left[1 + (n-1)p_{n-1}(x)h_n^m\left(\mathbf{K}\left((x - X_n)/h_n\right)\right)^{-1}\right]^{-1}.$$

Semi-recursive kernel type estimates of the production function

$$r_n(x) = \sum_{l=1}^{n} \frac{Y_l}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right) / \sum_{l=1}^{n} \frac{1}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right) \quad (4)$$

(see [6-8]). Weak and strong universal consistency of such estimates was investigated, for instance, in [9].

As above mentioned the studying of the asymptotic MSE for $r_n(x)$ has some difficulties due to the possible instability (for example, the denominator in (4) may be close to zero), and the theorems for MSE making use of the dominant sequences can not be applied [5, 10]. The problem can be resolved by using a piecewise smooth approximation. Therefore, similar to [10, 11], we use the estimate

$$\tilde{r}_n(x) = \frac{r_n(x)}{\left(1 + \delta_n |r_n(x)|^{\tau}\right)^{\rho}}, \quad (5)$$

where $\tau > 0$, $\rho > 0$, $\rho\tau \geq 1$, $(\delta_n) \downarrow 0$ as $n \to \infty$.

Similarly substitution estimates of $MP_j(x)$ and $MRTS_{ij}(x)$ are respectively

$$MP_{jn}(x) = \frac{\sum_{l=1}^{n} \frac{Y_l}{h_l^{m+1}}\mathbf{K}^{(1j)}\left(\frac{x - X_l}{h_l}\right)}{\sum_{l=1}^{n} \frac{1}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right)} - \frac{\sum_{l=1}^{n} \frac{Y_l}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right)\sum_{l=1}^{n} \frac{1}{h_l^{m+1}}\mathbf{K}^{(1j)}\left(\frac{x - X_l}{h_l}\right)}{\left(\sum_{l=1}^{n} \frac{1}{h_l^m}\mathbf{K}\left(\frac{x - X_l}{h_l}\right)\right)^2},$$

$$MRTS_{ji,n}(x) = \frac{MP_{jn}(x)}{MP_{in}(x)}, \tag{6}$$

and corresponding piecewise smooth approximations

$$\tilde{M}P_{jn}(x) = \frac{MP_{jn}(x)}{\left(1+\delta_n \mid MP_{jn}(x)\mid^{\tau}\right)^{\rho}},\ \tilde{M}RTS_{ji,n}(x) = \frac{MRTS_{ji,n}(x)}{\left(1+\delta_n \mid MRTS_{ji,n}(x)\mid^{\tau}\right)^{\rho}}. \tag{7}$$

Thus, in general the substitution estimate of $J(x)$ is $J_n(x) = H\left(\left\{a_n^{(rj)}(x)\right\}\right) = H\left(A_n\right)$, and its piecewise smooth approximation is $\tilde{J}_n(x) = \tilde{H}\left(A_n\right) = \dfrac{J_n(x)}{\left(1+\delta_n \mid J_n(x)\mid^{\tau}\right)^{\rho}}$.

As shown in [1], the rate of mean square convergence of estimates (4)−(7) depends on parameter ν characterizing the kernel:

$$T_j = \int_R u^j K(u)du = 0,\ j = 1,...,\nu-1,\quad T_\nu \neq 0,\ \int_R \left|u^\nu K(u)\right| du < \infty. \tag{8}$$

The main parts of the asymptotic MSE of the estimates (4)−(5) and (6)−(7) is of order $O\left(\left[\dfrac{1}{nh_n^m} + h_n^{2\nu}\right]^{3/2}\right)$ and $O\left(\left[\dfrac{1}{nh_n^m} + h_n^{2\nu+2}\right]^{3/2}\right)$ respectively. Under $\nu > 2$, it seems impossible to determine the MSE of the estimates [12, 13]. But it is shown in these papers that we can find the dominant sequence under $\nu = 2$ if, for example, $K(u) \geq 0$, and $Y < \infty$. Under $\nu > 2$ we can use the piecewise smooth approximation, and it is enough to take even $\nu \geq 4$ for obtaining the asymptotic MSE.

In (1) some variables of function $H(\cdot)$ may be absent, for example, all derivatives (r = 0 in (1)) or all basic functionals (r = 1 in (1)), otherwise r = 0,1.

Theorem. Let

1) functions $a_1(x) = \int_{R^+} yf(y \mid x)dy,\ a_2(x) = \int_{R^+} f(y \mid x)dy$ and their derivatives up to the order $\nu + \max(r)$ be bounded and continuous for any $x \in R^{m+}$, the function $\int_{R^+} y^4 f(y \mid x)dy$ be bounded on R^{m+}, the function $\int_{R^+} y^2 f(y \mid x)dy$ be continuous on R^{m+};

2) conditions (8) hold and $\sup_{u\in R}\left|K^{(r)}(u)\right| < \infty, \int_R \left|K^{\max(r)}(u)\right| du < \infty,$ if $\exists\, j : r = 1$ (i.e. there are derivatives in (1)) then, $\lim_{|u|\to\infty} K(u) = 0$, if the integer *r* take both the value 0 and the value 1 then in addition $K^{(1)}(u)$ be continuous on *R*;

3) $h_n + 1/nh_n^m \to 0$ as $n \to \infty$, and if $\exists\, j : r = 1$ in (1), then $h_n + 1/nh_n^{m+2} \to 0$;

4) the function $H(\cdot)$ be continuously differentiable up to the second order at the point $A = \left(a^{(rj)}(x)\right)$;

5) the sequence of functions $\left\{H(a_n^{(rj)}(x))\right\}$ for any possible sample values $Z_l = (X_l, Y_l),\ l = \overline{1,n}$ be dominated by a sequence of numbers $\left(C_0 d_n^\gamma\right),\ \left(d_n\right) \uparrow \infty, n \to \infty, 0 \leq \gamma \leq 1/4, C_0$ is a constant.

Then MSE of the estimates (1) can be written in the form

$$u^2\left(H\left(A_n\right)\right)=\sum_{i,p=1}^{2}\sum_{j,k=1}^{m}\sum_{r,q}H_{ijr}H_{pkq}\left(S_{-(m+2\max(r,q))}\frac{B_{i,p}^{((r,q)}}{nh_n^{m+r+q}}+S_\nu^2\omega_{i\nu}^{(rj)}(x)\omega_{p\nu}^{(qk)}(x)h_n^{2\nu}\right)+$$

$$+o\left(\left[\frac{1}{nh_n^{m+2\max(r,q)}}+h_n^{2\nu}\right]^{3/2}\right),$$

where

$$B_{i,p}^{(r,q)}=\int_R K^{(r)}(u)K^{(q)}(u)du\left(\int_R K^2(u)du\right)^{m-1}\int_R g_i(y)g_p(y)f(x,y)dy,$$

$$H_{ijr}=\partial H(A)/\partial a_i^{(rj)},\ \omega_{i\nu}^{(rj)}(x)=\frac{T_\nu}{\nu!}\sum_{l=1}^{m}\frac{\partial^{(\nu)}a_i^{(rj)}(x)}{\partial x_l^\nu},\ g_1=y,\ g_2=1.$$

Moreover, if the restriction 5) is replaced by $J(x)=H(a(x))\neq 0$ or $\tau\geq 4,\tau\in N^+$, then $u^2\left(\widetilde{H}\left(A_n\right)\right)=u^2\left(H\left(A_n\right)\right)$.

As above mentioned the restriction 5) of Theorem is the most problematical, and we don't need one when piecewise smooth approximations (5), (6) are used.

As to the second order production function's characteristics, they can be treated the same way because they came out of first order characteristics differentiation.

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