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PRICE CONTROL IN SELLING PERISHABLE GOODS

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The previous works of the author (1,2) were devoted to the problem of price control in selling perishables which are to be realized during a trading session. Below we consider price control in selling goods part of which may perish during a trading session.

1. Mathematical model of goods damage

Below we propose one of the possible mathematical models of goods damage.

Suppose the goods consist of separate elements (e.g. potatoes, fruit, etc.) which may perish in the storage period and which must be thrown out in the sale process.

Suppose there are $Q(t)$ such elements in a batch of goods. Imagine that at the interval $[t, t+\Delta t]$ with probability $p = \mu\Delta t + o(\Delta t)$ each element may perish. Let us designate through $\Delta Q(t)$ the number of perished elements at this interval. Then each element may be considered as experience in Bernoulli scheme, so $\Delta Q(t)$ is subordinated to binomial distribution

$$P\{\Delta Q\} = C_Q^{\Delta Q} p^{\Delta Q} (1-p)^{Q-\Delta Q}.$$

Hence we easily find statistical data of ΔQ. Using the features of binomial distribution we obtain

$$M\{\Delta Q\} = Qp = Q\mu\Delta t + o(\Delta t),$$

and in the diffusive approximation the process demolition factor $Q(t)$ is equal to $\mu Q(t)$.

Regarding $M\{\Delta Q^2\}$ we have

$$M\{\Delta Q^2\} = M\{\Delta Q\}^2 + D\{\Delta Q\} = Q^2\mu^2\Delta t^2 + Q\mu\Delta t(1-\mu\Delta t) + o(\mu\Delta t).$$

So in the diffusive approximation the process diffusion factor $Q(t)$ is equal to $\mu Q(t)$.

Below we will look at a more general model considering that the process diffusion factor $Q(t)$ is equal to $\sigma^2 Q(t)$.

Thus, if only the process of goods damage is considered, the quantity may be approximated by diffusion process

$$dQ(t) = -\mu Q(t)dt + \sqrt{\sigma^2 Q(t)}dw_t. \tag{1}$$

If we add the sale process here when the intensity of goods flow is equal to $\lambda(c(t))$, the process diffusive approximation $Q(t)$ will look like

$$dQ(t) = -(\mu Q(t) + a_1\lambda(c))dt + \sqrt{\sigma^2 Q(t) + a_2\lambda(c)}dw_t. \tag{2}$$

If price control in selling is used according to the formula

$$a_1\lambda(c) = \frac{Q(t)}{\varphi(t)},$$

the process diffusive approximation $Q(t)$ will look like

$$dQ(t) = -\left(\mu + \frac{1}{\varphi(t)}\right)Q(t)dt + \sqrt{\left(\sigma^2 + \frac{a_2}{a_1\varphi(t)}\right)Q(t)}dw_t. \tag{3}$$

It is the formula that we will use in the future.

2. First and second initial moments of the process *Q*(*t*)

Suppose $\overline{Q}(t) = M\{Q(t)\}$. Then by averaging (3) and considering that $M\{dw_t\} = 0$, we obtain the equation

$$\frac{d\overline{Q}(t)}{dt} = -\left(\mu + \frac{1}{\varphi(t)}\right)\overline{Q}(t), \tag{4}$$

which is to be solved with initial condition $\overline{Q}(0) = 0$.

Let us consider the law of price control with $\varphi(t) = \left(e^{\mu(CT-t)} - 1\right)/\mu$. The grounds for such choice $\varphi(t)$ are given in [3].

The solution of equation (4) looks as follows

$$\overline{Q}(t) = Q_0 \frac{e^{\mu(CT-t)} - 1}{e^{\mu CT} - 1}. \tag{5}$$

Now let us consider the process $Q^2(t)$. For the function $f(t, Q) = Q^2$ we have

$$\frac{\partial f}{\partial t} = 0;\ \frac{\partial f}{\partial Q} = 2Q;\ \frac{\partial^2 f}{\partial Q^2} = 2,$$

and according to Ito formula we see that the process $Q^2(t)$ satisfies the following stochastic differential equation

$$dQ^2(t) = \left[-2\left(\mu + \frac{1}{\varphi(t)}\right)Q^2(t) + \left(\sigma^2 + \frac{a_2}{a_1\varphi(t)}\right)Q(t)\right]dt + \sqrt{2\left(\sigma^2 + \frac{a_2}{a_1\varphi(t)}\right)Q^2(t)dw_t}. \tag{6}$$

Let us designate $Q_2(t) = M\{Q^2(t)\}$. Then by averaging (6) we obtain the following differential equation regarding $Q_2(t)$

$$\frac{dQ_2(t)}{dt} = -2\left(\mu + \frac{1}{\varphi(t)}\right)Q_2(t) + \left(\sigma^2 + \frac{a_2}{a_1\varphi(t)}\right)\overline{Q}(t), \tag{7}$$

which is to be solved with the initial condition $Q_2(0) = Q_0^2$ and with $\varphi(t) = \left(e^{\mu(CT-t)} - 1\right)/\mu$.

It may be shown that the solution of equation (7) looks like

$$Q_2(t) = Q_0^2\left(\frac{e^{\mu(CT-t)} - 1}{e^{\mu CT} - 1}\right)^2 + Q_0\left(\frac{e^{\mu(CT-t)} - 1}{e^{\mu CT} - 1}\right)^2 f(t), \tag{8}$$

where

$$f(t) = \left(\frac{\sigma^2}{\mu} + \frac{a_2}{a_1}\right)\left(e^{\mu CT} - 1\right)\ln\frac{e^{\mu CT} - 1}{e^{\mu CT} - e^{\mu t}} + \frac{a_2}{a_1}\frac{e^{\mu t} - 1}{e^{\mu CT} - e^{\mu t}}e^{\mu CT}. \tag{9}$$

The first item in this expression is $\overline{Q}^2(t)$, and the second item is dispersion $Q(t)$. It is easy to check that $D\{Q(0)\} = 0$ and $D\{Q(CT)\} = 0$, that is, the sale of all goods will be completed at moment CT.

3. Optimization of sale process

Now let us consider the problem of choosing optimal volume Q_0 of a batch of goods purchased for sale at wholesale price d and the problem of choosing optimal value of parameter C.

Suppose, as in the previous works of the author,

$$\lambda(c) = \lambda_0 - \lambda_1\frac{c - c_0}{c_0}.$$

Then the equality

$$a_1\lambda(c) = \frac{Q(t)}{\varphi(t)}$$

determines the dependence of price on time

$$c = c_0\left(1 + \frac{\lambda_0}{\lambda_1} - \frac{Q(t)}{a_1\lambda_1\varphi(t)}\right).$$

Then we obtain

$$a_1 c\lambda(c) = c_0\left(1 + \frac{\lambda_0}{\lambda_1} - \frac{Q(t)}{a_1\lambda_1\varphi(t)}\right)\frac{Q(t)}{\varphi(t)}.$$

The average value of profit from selling a batch of goods for time T is equal to

$$S = \int_0^T M\{a_1 c\lambda(c)\}dt = c_0\left(1 + \frac{\lambda_0}{\lambda_1}\right)\int_0^T \frac{\overline{Q}(t)}{\varphi(t)}dt - \frac{c_0}{a_1\lambda_1}\int_0^T \frac{Q_2(t)}{\varphi^2(t)}dt.$$

It may be shown that

$$\int_0^T \frac{\overline{Q}(t)}{\varphi(t)}dt = \frac{\mu T Q_0}{e^{\mu CT} - 1},$$

$$\int_0^T \frac{Q_2(t)}{\varphi^2(t)}dt = \frac{\mu^2 T Q_0^2}{(e^{\mu CT} - 1)^2} + \frac{\mu^2 Q_0}{(e^{\mu CT} - 1)^2}F(C),$$

where

$$F(C) = \int_0^T f(t)dt.$$

Now the average value of profit from selling our batch of goods may be put down as

$$P = S - dQ_0 =$$

$$= \left(c_0\left(1 + \frac{\lambda_0}{\lambda_1}\right)\frac{\mu T}{e^{\mu CT} - 1} - \frac{\mu^2 F(C)}{a_1\lambda_1(e^{\mu CT} - 1)^2} - d\right)Q_0 - \frac{c_0\mu^2 T}{a_1(e^{\mu CT} - 1)^2}Q_0^2, \qquad (10)$$

from which due to condition $\partial P/\partial Q_0 = 0$ we obtain the optimal volume of a batch of goods

$$Q_0 = \frac{a_1\lambda_1(e^{\mu CT} - 1)^2}{2\mu^2 T}\left[\left(1 + \frac{\lambda_0}{\lambda_1}\right)\frac{\mu T}{e^{\mu CT} - 1} - \frac{d}{c_0} - \frac{\mu^2 F(C)}{a_1\lambda_1(e^{\mu CT} - 1)^2}\right] \qquad (11)$$

and the average maximum profit

$$P_{\max} = \frac{a_1\lambda_1(e^{\mu CT} - 1)^2}{4\mu^2 T}\left[\left(1 + \frac{\lambda_0}{\lambda_1}\right)\frac{\mu T}{e^{\mu CT} - 1} - \frac{d}{c_0} - \frac{\mu^2 F(C)}{a_1\lambda_1(e^{\mu CT} - 1)^2}\right]^2. \qquad (12)$$

Now let us compute $F(C)$. For it we have

$$F(C) = \left(\frac{\sigma^2}{\mu} + \frac{a_2}{a_1}\right)(e^{\mu CT} - 1)\int_0^T \ln\left(\frac{e^{\mu CT} - 1}{e^{\mu CT} - e^{\mu t}}\right)dt + \frac{a_2}{a_1}e^{\mu CT}\int_0^T \frac{e^{\mu t} - 1}{e^{\mu CT} - e^{\mu t}}dt. \qquad (13)$$

The second integral after substituting variables $e^{\mu t} = z$ is easily computed

$$\int_0^T \frac{e^{\mu t} - 1}{e^{\mu CT} - e^{\mu t}}dt = \frac{1}{\mu}\int_1^{e^{\mu T}} \frac{z - 1}{(e^{\mu CT} - z)z}dz = \frac{1}{\mu}(e^{\mu CT} - 1)\ln\left(\frac{e^{\mu CT} - 1}{e^{\mu CT} - e^{\mu T}}\right) - CTe^{-\mu CT}.$$

As for the integral

$$\int_0^T \ln(e^{\mu CT} - e^{\mu t})dt = \frac{1}{\mu}\int_1^{e^{\mu T}} \frac{\ln(e^{\mu CT} - z)}{z}dz,$$

it is not expressed through elementary functions (it resembles integral logarithm). That is why the finding of optimal value C is possible only numerically.

4. Poisson approximation

Now to solve the problem under consideration let us look at another approximation which is not diffusive.

Let us designate through $Q(t)$ the amount of available goods at moment t. Then we have the correlation

$$Q(t+\Delta t)=Q(t)-\Delta Q(t), \tag{14}$$

where $\Delta Q(t)$ is the diminution of goods for time Δt. It may be presented as

$$\Delta Q(t)=\Delta Q_{исп}(t)+\Delta Q_{пр}(t), \tag{15}$$

where $\Delta Q_{исп}(t)$ is the amount of perished goods for time Δt and $\Delta Q_{пр}(t)$ is the amount of sold goods for the same period of time.

Regarding statistic characteristics of these quantities we take the following assumptions.

Let us consider that $\Delta Q_{исп}(t)$ is a random quantity with parameters

$$M\{\Delta Q_{исп}(t)\}=\mu Q(t)\Delta t+o(\Delta t),\ M\{\Delta Q^2_{исп}(t)\}=\sigma^2 Q(t)\Delta t+o(\Delta t). \tag{16}$$

Regarding $\Delta Q_{пр}(t)$ and considering Poisson approximation of the flow of purchases we accept the following model

$$\Delta Q_{пр}(t)=\begin{cases}0, & \text{with probability } 1-\lambda(c)\Delta t+o(\Delta t),\\ \xi, & \text{with probability } \lambda(c)\Delta t+o(\Delta t),\end{cases} \tag{17}$$

where ξ is the size of purchase. Let us consider that ξ is a random quantity with $M\{\xi\}=a_1$ and $M\{\xi^2\}=a_2$.

Now let us deduce an equation for $M\{Q(t)\}=\overline{Q}(t)$. By averaging $\Delta Q(t)$ according to factual purchases and sizes of purchases and taking into account the law of price control we obtain

$$M\{\Delta Q(t)\}=\left[\mu Q(t)+\frac{Q(t)}{\varphi(t)}\right]\Delta t+o(\Delta t).$$

So by averaging (14) we get

$$\frac{d\overline{Q}(t)}{dt}=-\left(\mu+\frac{1}{\varphi(t)}\right)\overline{Q}(t),$$

which coincides with equation (4) obtained in diffusive approximation.

Let us have $Q_2(t)=M\{Q^2(t)\}$. By squaring (14) we obtain

$$Q^2(t+\Delta t)=Q^2(t)-2Q(t)\Delta Q(t)+(\Delta Q(t))^2. \tag{18}$$

By averaging this expression we may obtain

$$\frac{dQ_2(t)}{dt}=-2\left(\mu+\frac{1}{\varphi(t)}\right)Q_2(t)+\left(\sigma^2+\frac{a_2}{a_1\varphi(t)}\right)\overline{Q}(t). \tag{19}$$

This equation coincides with equation (7). Thus the approximation under consideration gives the same result as diffusive approximation does.

Literature

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