NUMERICAL ANALYSIS OF METHODS OF SOLVING OPTIMAL CONTROL PROBLEMS WITH NON-FIXED TIME

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Methods of numerical solving problems of optimal control by objects with distributed and concentrated parameters with non-fixed (optimized) termination time of a process are investigated and compared in the work. Particular case of this class of problems is a quick-action problem – the quickest transfer of the object from one state to another.

1. For a start consider a problem of optimal control by an object with concentrated parameters in the following statement:

$$\dot{x}(t) = f(x,u), \quad t \in (0,T], \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$J(u,T) = \int_0^T f^0(x,u)dt + \Phi(x(T),T) \to \min_{T,\, u\in U}, \tag{3}$$

where $x(t) \in E^n$ is a phase trajectory, $u(t) \in E^r$ is a control action, $U \subset E^r$ is a set of admissible values of the control, $T > 0$ is the optimized termination time of the process, the given vector-function $f(.,.)$, scalar functions $f^0(.,.)$, $\Phi(.,.)$ are continuously differentiable on their arguments.

In order to solve problem (1)-(3) the following two approaches are used in practice. According to the first approach first order methods, for instance gradient projection method in the space of control and time T, are used:

$$u^{k+1}(t) = \Pr_U (u^k(t) - \alpha_k \cdot grad_u J(u^k, T^k)), \tag{4}$$

$$T^{k+1} = T^k - \alpha_k \frac{\partial J(u^k, T^k)}{\partial T}, \tag{5}$$

where the gradient of the functional is determined by the following formulas [1-5]:

$$grad_u J(u,T) = -\frac{\partial f^0(x,u)}{\partial u} + \frac{\partial f(x,u)}{\partial u}\psi(t), \tag{6}$$

$$\frac{\partial J(u,T)}{\partial T} = f(x(T),u(T)) \cdot \psi(T) + \frac{\partial \Phi(x(T),T)}{\partial T}, \tag{7}$$

$$\dot{\psi}(t) = -\frac{\partial f(x,u)}{\partial x}\psi(t) + \frac{\partial f^0(x,u)}{\partial x}, \tag{8}$$

$$\psi(T) = -\frac{\partial \Phi(x(T),T)}{\partial x}, \tag{9}$$

where α_k is a step of one-dimensional minimization, chosen from the condition:

$$\alpha_k = \arg\min_{\alpha \ge 0} J\left(\Pr_U \left(u^k(t) - \alpha \cdot grad_u J(u^k, T^k)\right), T^k - \alpha \frac{\partial J(u^k, T^k)}{\partial T} \right).$$

Thus while using procedures (4), (5) for solving problem (1)-(3), the control actions $u^k(t)$ and the termination time T^k of the process are changed simultaneously.

It is important to note the following main drawback of such approach at $T^{k+1} > T^k$, consisting in the uncertainty of the value of the control $u^k(t)$ on the interval $\left[T^k, T^{k+1}\right]$ (from (5) it is clear that in case if $\dfrac{\partial J(u^k, T^k)}{\partial T} > 0$ then $T^{k+1} > T^k$).

The latter approach to the solution to problem (1)-(3) consists in using two-level optimization:

$$\min_{T,\, u(t)\in U} J(u,T) = \min_{T>0} \min_{u(t)\in U} J(u,T).$$

In this case for outer minimization on T some of the methods of one-dimensional optimization (for instance method of golden section or bisection) is used, and at every fixed (calculated by method of one-dimensional optimization) value $\overline{T}$ method of gradient projection in the space of only the control (4) for the minimization of the functional

$$\overline{J}_{\overline{T}}(u) = J(u, \overline{T}) \to \min_{u\in U}$$

is used.

Note that in spite of the fact that both these approaches to the solution to problem (1)-(3) are well-known, there have not been made a comparison between them, and investigators, as a rule, adhere to one of these approaches[4,8].

The results of numerical experiments will be given, analysis and comparison will be conducted, and recommendations on using both of the approaches will be made in the report.

2. As a case in point in controlling systems with distributed parameters consider the following problem of control by heating of a bar with non-fixed termination time of the process [6,7]:

$$\frac{\partial v(x,t)}{\partial t} = a^2 \frac{\partial^2 v(x,t)}{\partial x^2} + u(x,t), \quad x \in (0,l), \quad t \in (0,T], \tag{10}$$

$$v(x,0) = \varphi(x), \tag{11}$$

$$v(0,t) = \varphi_0(t), \quad v(l,t) = \varphi_1(t), \tag{12}$$

$$J(u(x,t),T) = \int_0^T\int_0^l f^0(u,v)\,dxdt + \int_0^l f^1(u(x,T), v(x,T))\,dx + \Phi(T) \to \min_{T,\, u(x,t)\in U} \tag{13}$$

where $v(x,t)$ is the temperature of the bar at the point $x \in (0,l)$ at the moment of time $t \in (0,T]$; $u(x,t)$ is a control determining the temperature of distributed sources, U is a set of admissible values of the control, $\varphi(x)$, $\varphi_0(t)$, $\varphi_1(t)$, $f^0(u,v)$, $f^1(u,v)$ are the given functions, l is the length of the bar.

For numerical solving problem (10)-(13) the two approaches can be used too.

The first approach is based on gradient procedure in the space of control $u(x,t)$ and time T:

$$u^{k+1}(t) = P_r(u^k(x,t) - \alpha_k \cdot grad_u J(u^k, T^k)), \tag{14}$$

$$T^{k+1} = T^k - \alpha_k \frac{\partial J(u^k(x,t), T^k)}{\partial T}, \tag{15}$$

where the gradient of the functional is determined by the formulas:

$$grad\ J(u,T) = \psi(x,t), \tag{16}$$

$$\frac{\partial J(u(x,t),T)}{\partial T} = \int_0^l (a^2 v''_{xx}(x,T) + u(x,T)) \cdot \psi(x,T)dx + \Phi'_T(T) =$$

$$= \int_0^l v'_t(x,T) \cdot \psi(x,T)dx + \Phi'_T(T), \tag{17}$$

$$\frac{\partial \psi(x,t)}{\partial t} = -a^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial f^0(u,v)}{\partial v}, \tag{18}$$

$$\psi(x,T) = -\frac{\partial f^1(u(x,T),v(x,T))}{\partial u}, \tag{19}$$

$$\psi(0,t) = \psi(l,t) = 0. \tag{20}$$

Relations (18)-(20) define the adjoint problem.

The other approach, which is similar to the one stated in item 1, uses two-level minimization. At the upper (outer) level one-dimensional optimization on the termination time T of the process with the use of some one-dimensional method is carried out. At each given $\overline{T}$ procedure (14) for solving optimal control problem (11)-(13) with fixed termination time of the process of heating is used.

3. One can consider a case in problem (10)-(13), when the value of T is given, and the length l of the bar is not given but optimized. In this case it is also possible to use the two approaches, when optimization is carried out on the length of the bar separately, i.e.

$$\min_{l,\, u(x,t)\in U} J(u(x,t),l) = \min_{l>0} \min_{u(x,t)\in U} J(u(x,t),l),$$

$$\overline{J}_{\bar{l}}(u^*(x,t)) = \min_{u(x,t)\in U} J(u(x,t),l),$$

and at every given length of the bar a standard optimal control problem with fixed time and length is solved with the use of procedure (14).

It is possible to carry out optimization simultaneously on the control $u(x,t)$, time T and length l. For that it is necessary to use the formula:

$$\frac{\partial J(u,l)}{\partial l} = \int_0^T v'_x(l,t) \cdot \psi'_x(l,t)dt, \tag{21}$$

which one can obtain using method of variations.

Comparison of both the approaches stated above for numerical solving optimal control problems with non-fixed time was made. Numerical experiments showed on the whole big effectiveness of the first approach both for controlling an object with concentrated parameters, and an object with distributed parameters.

4. A two-dimensional case of the problem (10)-(13) concerning the heating of a rectangular plate with optimized sizes, terminated time and regimes of heating process, was also investigated:

$$\frac{\partial u}{\partial t} = a^2 \Delta u + u(x,t), \qquad x \in \Omega \subset E^2, t \in (0,T], \tag{22}$$

$$\Omega = \{x \in E^2;\ x_1 \in (0,l_1), \quad x_2 \in (0,l_2)\},$$

$$u(x,0) = \varphi(x), \qquad x \in \Omega, \tag{23}$$

$$u(x,t) = \varphi_1(x), \qquad x \in \Gamma = \overline{\Omega}/\Omega, \tag{24}$$

$$J(u(x,t),T,l_1,l_2) \to \min, \tag{25}$$

where l_1, l_2 determine the boundary of the plate, J is any given functional, for example, such as in (13).

In practice, the problem (22)-(25) arises, in general, as an inverse problem of parametrical identification of heating process.

Formulas for the gradient of the functional on $T, l,$ similar to (21), were obtained:

$$\frac{dJ(u,T,l)}{dl_1} = \int_0^T \int_0^{l_2} v'_{x_1}(l_1, x_2, t)\psi'_{x_1}(l_1, x_2, t)dx_2 dt ,$$

$$\frac{dJ(u,T,l)}{dl_2} = \int_0^T \int_0^{l_1} v'_{x_2}(x_1, l_2, t)\psi'_{x_2}(x_1, l_2, t)dx_1 dt ,$$

$$\frac{dJ(u,T,l)}{dT} = \int_0^{l_2} \int_0^{l_1} v'_t(x_1, x_2, T)\psi(x_1, x_2, T)dx_1 dx_2 + \Phi'_T(T) ,$$

that allow to formulate the necessary conditions of the first order optimality.

Note that the considered problem, as well as the problem (10)-(13), can be related to problems of optimization of a domain form.

The comparison of the approaches to the numerical solution to the problems (22)-(25), when one-level optimization and separate optimization on $T, l(t)$ and $u(x,t)$ are used, was made.

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