*The Second International Conference "Problems of Cybernetics and Informatics"*
*September 10-12, 2008, Baku, Azerbaijan. Section #5 "Control and Optimization"*
*www.pci2008.science.az/5/25.pdf*

# COMPLETELY CONTROLLABLE QUANTUM SYSTEMS

**Grigory Giorgadze**

Tbisisi State Universiry, Institute of Cybernetics, Tbilisi, Georgia, *giorgadze@rmi.acnet.ge*

We consider second order Fuchsian systems as Schrödinger type equation and shown that some values of parameters this systems describes completely controllable quantum systems [1]. J Duistermaat introduced th econcept of monodromy in the case of integrable Hamiltonian systems of two degree of freedom. Since that, the monodromy has been found and analyzed for several integrable systems of classical mechanics and analytical solvable model of quantum dynamics. First who considered such approach were L.Landau, N.Rosen and C.Zener. This subsequently has been generalized by several authors. Geometrically, monodromy describes the global twisting of family (bundle) of invariant 2-tori parametrized by (over) the circle of the regular values of energy momentum map of the integrable system. Its presence is indicated by the existence of a singular fiber of the energy momentum map which is topologically a "pinched torus". Loosely speaking, if an integrable system has monodromy then it is impossible to label the tori in a unique way by values of the action. Since invariant tori are at the foundations of semiclassical Einshtein-Brillouin-Kramers quantization of integrable systems, monodromy should manifest itself in the corresponding quantum systems. Because monodromy is quite common in the classical integrable systems of two degree freedom, it should have many important physical implementations in quantum mechanics. Cyclic evolution of external parameters after leads to evolution involving a phase depending only on the geometry of the path traversed in parameter space. The natural mathematical language for the geometric phases is the theory of fiber bundles. There one defines a phase, known as holonomy, that depends on the geometry of a loop and is independent of any choice of coordinates. After the paper M.Berry, this additional phase is called geometric. The presence of geometric phase in quantum system and its properties have been used by P.Zanardy and M.Rasseti to construct a new model for quantum computing. The irreducibility of the connection in holonomic quantum computing guarantees the controllability of process of computation. In [2] we considered model of quantum computations based on the monodromy representation of the Fuchsian system and in [3] shown, that this system describes dynamic of quantum system. This reduction for the two level quantum system given below.

Let $M_1, M_2, M_3, M_4$ such matrices, such that $E_j = (1/2\pi i)\log M_j$, $j = 1,...,4$ and $M_1, M_2, M_3$ make a basis of the Lie algebra of $SU(2)$ and satisfy the condition $M_1 M_2 M_3 M_4 = 1$. Let $s_1, s_2, s_3, s_4 \in CP^1$ any points. Denote by $X_4 = CP^1 \setminus \{s_1, s_2, s_3, s_4\}$ and by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ the generators of $\pi_1(X_4, x_0)$ with relation $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$. Consider the representation $\rho$ of the fundamental group $\pi_1(X_4, x_0)$ in $SU(2)$ defined by the correspondence

$$\rho : \pi_1(X_4, x_0) \to SU(2),\ \gamma_j \mapsto M_j.$$

Then for $\rho$ there exits the system of differential equations of Fuchs type

$$dF(z) = \left( \frac{A_1}{z - s_1} dz + \frac{A_2}{z - s_2} dz + \frac{A_3}{z - s_3} dz + \frac{A_4}{z - s_4} dz \right) F(z)$$

whose monodromy representation coincides of $\rho$. The representation $\rho$ induces two-dimensional vector bundle $E \to CP^1$ with meromorphic connection

$$\nabla = \sum_{j=1}^{4} \frac{A_j}{z - s_j} dz$$

and Chern number $c_1(E) = \sum_{j=1}^{4} trE_j$ . The solution space $L$ of the system is two dimensional vector space and $\pi_1(X_4, x_0)$ act on the $L$ and any unitary operator shell be obtain in this way.

Fuchsian equation of second order on the complex plane having three singular points is known as the Riemann equation. Coefficients of this equation are uniquely defined by characteristic roots and locations of the singular points. Second order equations with more than three singular points do not have this property anymore.

The hypergeomertic equation

$$z(z-1)\frac{d^2 f(z)}{dz^2} + (\gamma - (1+\alpha+\beta))\frac{df(z)}{dz} - \alpha\beta f(z) = 0$$

is a Schrödinger type equation

$$i\frac{\partial\psi(t)}{\partial t} = H(t)\psi(t) \qquad (1)$$

where $\psi(t) = (\psi_1(t), \psi_2(t))$ and the time dependent Hamiltonian $H(t)$ has the form

$$H(t) = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix}, \qquad (2)$$

where $\varepsilon(t) = E_0 \sec h(t/T) + E_1 \tanh(t/T)$, $V(t) = V_0$ and $E_0, E_1, T, V_0$ are constants.

**Theorem 1.** The system described by Schrödinger equation (1) with Hamiltonian (2) defines completely controllable quantum system iff satisfying the following conditions:

1. $iT(-E_1 + \sqrt{E_1^2 + V_0^3}) \notin Z$ ,
2. $iT(-E_1 - \sqrt{E_1^2 + V_0^3}) \notin Z$ ,
3. $\frac{1}{2} + E_0 T + iT\sqrt{E_1^2 + V_0^3}) \notin Z$ ,
4. $\frac{1}{2} + E_0 T - 2iE_1 T + iT\sqrt{E_1^2 + V_0^3}) \notin Z$ ,
5. $\frac{1}{2} - E_0 T + iT\sqrt{E_1^2 + V_0^3} \notin Z$ ,
6. $\frac{1}{2} - E_0 T - iT\sqrt{E_1^2 + V_0^3} \notin Z$

Completely integrable Fuchsian systems are known in mathematical physics as the Calogero-Maser-Sutherland systems. For example, in a particular cases the trigonometric Calogero-Maser-Sutherland system describes the dynamics of a quantum mechanical system of $n$ particles moving on the real line under the influence of a pair potential that is proportional to the inverse square of the hyperbolic sine of the distance of the particles. It is also known, that the trigonometric Calogero-Maser-Sutherland system is obtained as a limit of the elliptic Calogero-Maser-Sutherland system.

In [4] the authors classified integrable models of quantum mechanics which are invariant under the action of a Weil group with certain assumptions. For the case of $B_N (N \geq 3)$, the generic models coincides with the $BC_N$ Inozemtsev model. In [5] is found that the eigenvalue problem for the Hamiltonian of the $BC_1$ (one particle) Inozemtsev model is transformed to the Heun equation. The Heun equation is a standart form of Fuchsian differential equation with four regular singularities. It is written as

$$\left(\left(\frac{d}{dw}\right)^2 + \left(\frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\varepsilon}{w-t}\right)\frac{d}{dw} + \frac{\alpha\beta w - q}{w(w-1)(w-t)}\right)g(w) = 0$$

with the condition $\alpha + \beta + 1 = \gamma + \delta + \varepsilon$ . It is easy to transform an arbitrary Fuchsian differential equation with four regular singularities into the Heun equation.

A nontrivial Heun equation can be transformed to a hypergeometric equation by the rational substitution iff this substitution id rational function and satisfies so called PHH-properties [6].

**Theorem 2**. If Heun equation satisfies the PHH-properties, then it defines a completely controllable quantum system.

Indeed, if the condition of theorem are satisfied, then the property of complete controllability coincides with irreducibility of the monodromy representation of hypergeometric equation, which obtained from PHH-reduction. The condition of irreducibility of the monodromy representation of the hypergeometric equation is given by exponents of equation.

## Literature

1. A.A.Agrachev, Yu.L.Sachkov. Control theory from the geometric viewpoint. Springer-Verlag (2004), 412 p.
2. G.K.Giorgadze. Monodromy approach to quantum computing. International Journal of Modern Physics. (2002), vol.16, No 30, pp. 4593-4605
3. G.K.Giorgadze, G.N. Khimshiashvili. On Schredinger Equations of Okubo Type. Journal of Dynamical and Control Systems (2004), vol.10, N. 2, pp.171-186
4. H.Ochiai, T.Oshima, H.Sekiguchi. Computing families of symmetric differential equations. Proc.Japan Acad. (1994), vol.70, pp.62-66
5. T.Oshima, H.Sekiguchi. Computing families of the differential operators invariants under the actions of a Weil group. J.Math.Sci.Univ.Tokyo (1995), vol.2, pp.1-75
6. R.Maier. Transforming the Heun equation to the hypergeometric equation (2002) ArXiv: math.CA/0203264