POSTOPTIMAL ANALYSIS OF LEXICOGRAPHIC COMBINATORIAL MINIMAX PROBLEMS

Vladimir Emelichev[1], Evgeny Gurevsky[2]

Belarusian State University, Minsk, Belarus
[1]*emelichev@bsu.by*, [2]*evgeny_gurevsky@tut.by*

Many problems of design, planning and management in technical and organizational systems have a pronounced multicriteria character. Multiobjective models that are appeared in these cases are reduced to the choice of the "best" (in a certain sense) values of variable parameters from some discrete aggregate of the given quantities. Therefore recent interest of mathematicians to multicriteria discrete optimization problems keeps very high which is confirmed by the intensive publishing activity (see, e.g., bibliography in [1], which contains 234 references, and monographs [2–4]). One of the important directions of study such problems is stability analysis of solutions under perturbations of the initial data.

Stability theory is an integral part of traditional categories of mathematics. J. Hadamard has included the stability to the concept of correct mathematical problem along with the existence and the uniqueness of solution. Stability problem in optimization appears in the case when the set of feasible solutions of the problem and (or) the objective function depend on parameters for which are only known the domain of values. Availability of such parameters in optimization models is determined by various kinds of uncertainty such as lack of input data, inadequacy of mathematical models to real processes, rounding off, calculating errors and other factors. In these conditions it is very important to choose the classes of problems in which any "small" changes of input parameters correspond to "small" changes of output results. It is evident that any optimization problem that is appeared in practice can not be correctly formulated and solved without using of stability theory results.

In this paper we consider a vector variant of a classical extremal combinatorial bottleneck problem, i.e. minimax problem on a system of nonempty subsets of a finite set. Necessary and sufficient conditions of five well-known (see, e.g., [3, 5, 6]) types of stability which differently describe the behavior of lexicographic set under independent changes of parameters of partial criteria are obtained. Analogues results for vector minimax problems with Pareto choice function were obtained earlier in [6].

Let $N_m = \{1, 2, \ldots, m\}$, $m \geq 2$, be the given set and $T \subseteq 2^{N_m} \setminus \{\varnothing\}$ be a some system of nonempty subsets of N_m, $|T| \geq 2$. Elements of set T are called trajectories. The components of vector-function $f(t, A) = (f_1(t, A), f_2(t, A), \ldots, f_n(t, A))$, $n \geq 1$, defined on T are minimax criteria

$$f_i(t, A) = \max_{j \in t} a_{ij} \to \min_{t \in T}, \ i \in N_n,$$

where $A = [a_{ij}] \in \mathbf{R}^{n \times m}$.

Under the n-criteria trajectory problem $Z^n(A)$, $n \geq 1$, we understand the problem of finding lexicographic set

$$L^n(A) = \{t \in T : \forall t' \in T \ \ (t \overline{\underset{A}{\succ}} t')\},$$

where $\overline{\underset{A}{\succ}}$ is the negation of binary lexicographic relation $\underset{A}{\succ}$, which is defined on the set T by the formula:

$$t \underset{A}{\succ} t' \Leftrightarrow \exists k \in N_n \ \ (f_k(t, A) > f_k(t', A) \ \& \ k = \min\{i \in N_n : f_i(t, A) \neq f_i(t', A)\}).$$

Note, that many classical extremal problems on graphs (salesman problem, shortest-path problem, spanning tree problem and others), different problems of scheduling theory and boolean programming problems are putted in scheme of scalar (single criterion) combinatorial problems (with linear, minimax and other kinds of criteria) [2, 4, 6–8].

Problem $Z^n(A)$ is called stable, if

$$\exists \varepsilon > 0 \ \ \forall B \in \Xi(\varepsilon) \ \ (L^n(A+B) \subseteq L^n(A)),$$

quasi-stable, if

$$\exists \varepsilon > 0 \ \ \forall B \in \Xi(\varepsilon) \ \ (L^n(A) \subseteq L^n(A+B)),$$

strongly stable, if

$$\exists \varepsilon > 0 \ \ \forall B \in \Xi(\varepsilon) \ \ (L^n(A) \cap L^n(A+B) \neq \varnothing),$$

strongly quasi-stable, if

$$\exists \varepsilon > 0 \ \ \exists t \in L^n(A) \ \ \forall B \in \Xi(\varepsilon) \ \ (t \in L^n(A+B)),$$

unalterable, if

$$\exists \varepsilon > 0 \ \ \forall B \in \Xi(\varepsilon) \ \ (L^n(A) = L^n(A+B)),$$

where $\Xi(\varepsilon) = \{B \in \mathbf{R}^{n \times m} : \| B \| < \varepsilon\}$ is the set of perturbing matrices, $\| B \| = \max\{| b_{ij} | : (i, j) \in N_n \times N_m\}$, $B = [b_{ij}]$.

Note, that stability and quasi-stability of $Z^n(A)$ can be interpreted as a discrete analogues of the Hausdorff upper and lower semi-continuity properties (respectively) of the point-to-set mapping [2, 3, 9], which determines the lexicographic choice function:

$$L^n : \mathbf{R}^{n \times m} \to 2^T.$$

But most results of stability investigating touch in general on vector linear and quadratic integer programming problems [2, 3, 10–12]. In this paper the five types of stability listed above are first studied for the problem of sequential optimization with minimax criteria. Let us introduce a several denotations.

For any index $i \in N_n$ we set a binary relation on T as follows:

$$t \underset{i}{\text{ђ}} t' \Leftrightarrow N_i(t) \supseteq N_i(t'),$$

where

$$N_i(t) = \{ j \in t : a_{ij} = f_i(t, A)\}.$$

Suppose

$$L_i^n(A) = \operatorname{Arg\,min}\{ f_i(t, A) : t \in L_{i-1}^n(A)\}, i \in N_n, \text{ where } L_0^n(A) = T.$$

$$M(t) = \{ i \in N_n : t \in L_i^n(A)\},$$

$$U^n(A) = \{ t \in \overline{L^n}(A) \cap L_1^n(A) : \forall t' \in L^n(A) \ \ \exists k \in M(t) \ \ (t \underset{k}{\overline{\text{ђ}}} t')\},$$

$$V^n(A) = \{ t' \in L^n(A) : \forall i \in N_n \ \ \forall t \in L_i^n(A) \ \ (t \underset{i}{\text{ђ}} t')\}.$$

Then $L_n^n(A) = L^n(A)$. In these denotations the following necessary and sufficient conditions of five types of stability formulated above are true.

Theorem 1. *For problem $Z^n(A)$, $n \geq 1$, the following statements are equivalent:*

(i) $Z^n(A)$ is stable,

(ii) $Z^n(A)$ is strongly stable,

(iii) $U^n(A) = \varnothing$.

Theorem 2. *Problem $Z^n(A)$, $n \geq 1$, is quasi-stable if and only if $L^n(A) = V^n(A)$.*

Theorem 3. *Problem $Z^n(A)$, $n \geq 1$, is strongly quasi-stable if and only if $V^n(A) \neq \varnothing$.*

Theorems 1 and 2 imply

Theorem 4. *Problem $Z^n(A)$, $n \geq 1$, is unalterable if and only if $U^n(A) = \varnothing$ and $L^n(A) = V^n(A)$.*

Literature

1. M. Ehrgott, X. Gandibleux. A survey and annotated bibliography of multiobjective combinatorial optimization // OR Spectrum. 2000. V. 22. N. 4. PP. 425–460.
2. I.V. Sergienko, L.N. Kozeratskaya, T.T. Lebedeva. Stability Investigation and Parametric Analysis of Discrete Optimization Problems. (In Russian) "Naukova Dumka", Kiev (1995).
3. I.V. Sergienko, V.P. Shilo. Discrete Optimization Problems: Issues, Solution Methods and Investigations. (In Russian) "Naukova Dumka", Kiev (2003).
4. Yu.N. Sotskov, N.Yu. Sotskova. Scheduling Theory. Systems with Uncertain Numerical Parameters. (In Russian) "UIIP of NAS of Belarus", Minsk (2004).
5. V.A. Emelichev, E. Girlich, Yu.V. Nikulin, D.P. Podkopaev. Stability and regularization of vector problems of integer linear programming // Optimization. 2002. V. 51. N 4. PP. 645–676.
6. V.A. Emelichev, K.G. Kuzmin. Stability criteria in a terms of binary relations for the vector combinatorial bottleneck problems // Cybernetics and Systems Analysis. 2008. V. 44. N 3.
7. V.A. Emelichev, M.K. Kravtsov. Combinatorial problems of vector optimization // Discrete Mathematics and Applications. 1995. V. 5. N 2. PP. 93–106.
8. Yu.N. Sotskov, V.K. Leontev, E.N. Gordeev. Some concepts of stability analysis in combinatorial optimization // Discrete Applied Mathematics. 1995. V. 58. N 2. PP. 169–190.
9. T. Tanino. Sensitivity analysis in multiobjective optimization // Journal of Optimization Theory and Applications. 1988. V. 56. N 3. PP. 479–499.
10. T.T. Lebedeva, T.I. Sergienko. Comparative analysis of different types of stability with respect to constraints of a vector integer-optimization problem // Cybernetics and Systems Analysis. 2004. V. 40. N 1. PP. 52–57.
11. T.T. Lebedeva, N.V. Semenova, T.I. Sergienko. Stability of vector problems of integer optimization: relationship with the stability of sets of optimal and nonoptimal solutions // Cybernetics and Systems Analysis. 2005. V. 41. N 4. PP. 551–558.
12. T.T. Lebedeva, T.I. Sergienko. Stability of a vector integer quadratic programming problem with respect to vector criterion and constraints // Cybernetics and Systems Analysis. 2006. V. 42. N 5. PP. 667–674.