NON-LOCAL OPTIMAL CONTROL PROBLEM FOR MANJERON GENERALIZED EQUATION WITH NON-SMOOTH COEFFICIENTS UNDER SEPARATED MULTI-POINT INITIAL BOUNDARY CONDITIONS

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Such an optimal control problem is investigated with the help of a new variant of the increment method. This method essentially uses the notion of adjoint equation of integral form and allows to cover the case when the coefficients of the equation are, generally speaking non-smooth functions.

Necessary and sufficient optimality conditions were studied in the monographs of e.g. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishenko [1], R.Bellman [2] and others. We notice also the work of A.I.Egorov [3], K.T.Akhmedov and S.S. Akhiyev [4] and etc. in which different classes of optimal control problems were studied.

Some classes of optimal processes related with non-local boundary value problems are studied in the papers [5,6], as well.

Let a controlled object be described by the Manjeron equation

$$\begin{aligned}(V_{2,2}u)(x) \equiv D_1^2 D_2^2 u(x) + a_{2,1}(x) D_1^2 D_2 u(x) + a_{1,2}(x) D_1 D_2^2 u(x) + a_{2,0}(x) D_1^2 u(x) + \\ + a_{0,2}(x) D_2^2 u(x) + \sum_{i=0}^{1}\sum_{j=0}^{1} a_{i,j}(x) D_1^i D_2^j u(x) = \varphi(x, \upsilon(x)), \quad x = (x_1, x_2),\end{aligned} \tag{1}$$

under the following separated multi-point initial boundary conditions

$$\begin{cases} V_{0,0}^{(k)} u \equiv u(\tau_k, \xi_k) = \varphi_{0,0}^{(k)} \in R; \\ V_{1,0}^{(k)} u \equiv D_1 u(\tau_k, \xi_k) = \varphi_{1,0}^{(k)} \in R; \\ V_{0,1}^{(k)} u \equiv D_2 u(\tau_k, \xi_k) = \varphi_{0,1}^{(k)} \in R; \\ V_{1,1}^{(k)} u \equiv D_1 D_2 u(\tau_k, \xi_k) = \varphi_{1,1}^{(k)} \in R; \\ (V_{2,0}^{(k)} u)(x_1) \equiv D_1^2 u(x_1, \xi_k) = \varphi_{2,0}^{(k)}(x_1) \in L_p(G_1); \\ (V_{2,1}^{(k)} u)(x_1) \equiv D_1^2 D_2 u(x_1, \xi_k) = \varphi_{2,1}^{(k)}(x_1) \in L_p(G_1); \\ (V_{0,2}^{(k)} u)(x_2) \equiv D_2^2 u(\tau_k, x_2) = \varphi_{0,2}^{(k)}(x_2) \in L_p(G_2); \\ (V_{1,2}^{(k)} u)(x_2) \equiv D_1 D_2^2 u(\tau_k, x_2) = \varphi_{1,2}^{(k)}(x_2) \in L_p(G_2); \end{cases} \tag{2}$$

where $\varphi_{i,j}^{(k)}$, $i, j = \overline{0,1}$ are the given constants, the remaining $\varphi_{i,j}^{(k)}$ are the given measurable functions; $D_k = \partial/\partial x_k$, $(k = \overline{1,2})$ is a generalized differentiation operator in S.L.Sobolev's sense. Besides, the given above $a_{i,j}(x)$ are measurable functions on $G = G_1 \times G_2$; $G_i = (0, h_i)$, $i = \overline{1,2}$ and satisfy only the following conditions:

$$a_{i,j}(x) \in L_p(G), \quad i, j = \overline{0,1}; \quad a_{2,j}(x) \in L_{\infty,p}^{x_1,x_2}(G), \quad j = \overline{0,1}; \; a_{i,2}(x) \in L_{p,\infty}^{x_1,x_2}(G), \quad i = \overline{0,1};$$

Notice that here we assume, (τ_k, ξ_k), $(k = \overline{1,N})$ are the fixed points from $\overline{G}$; $\varphi(x, \upsilon(x))$ is the given function on $G \times R^r$ satisfying Caratheodory conditions in $G \times R^r$, (i.e. $\varphi(x, \upsilon(x))$ is measurable with respect to x on G for all given $\upsilon \in R^r$ and continuous with respect to υ on R^r almost for all given $x \in G)$ and for positive number $\delta > 0$ there exists such a function

$\varphi_\delta^0(x)\in L_p(G)$ that $|\varphi(x,\upsilon(x))|\le\varphi_\delta^0(x)$ almost for all $x\in G$ and all $\upsilon\in R^r$ for which $\|\upsilon\|=\sum_{i=1}^r|\upsilon_i|\le\delta$; $\upsilon(x)=(\upsilon_1(x),...,\upsilon_r(x))$ is r-dimensional controlling vector-function.

Let the vector-function $\upsilon(x)$ be measurable and bounded on G and almost at all points $x\in G$ it accepts its values from some given set $\Omega\subset R^r$. Then this vector-function is said to be admissible control. A set of admissible control denoted by Ω_∂.

Now, let's consider the following non-local optimal control problem: find the admissible controls is $\upsilon(x)$ from Ω_∂, for which the solution of non-local problem (1)-(2) in the S.L.Sobolev's space

$$u\in W_p^{(2,2)}(G)\equiv\left\{u(x):D_1^iD_2^ju(x)\in L_p(G),\ i,j=\overline{0,2},\ \right\},\quad(1\le p\le\infty)$$

delivers the least value to the multi-point functional

$$S(\upsilon)=\sum_{k=1}^N\left[\alpha_ku\left(\tau_k,\lambda_k^{(0)}\right)+\beta_ku\left(\mu_k^{(0)},\xi_k\right)+\gamma_ku\left(\mu_k^{(0)},\lambda_k^{(0)}\right)\right]\to\min,\tag{3}$$

where $\left(\mu_k^{(0)},\lambda_k^{(0)}\right)\in\overline{G}$ are the given points; $\alpha_k\in R$, $\beta_k\in R$ and $\gamma_k\in R$ are the given numbers; N is a natural number.

In order to obtain necessary and suffucients conditions of optimality at first we find the increment of the functional (3). Let $\upsilon(x)$ and $\upsilon(x)+\Delta\upsilon(x)$ be different admissible controls, $u(x)$ and $u(x)+\Delta u(x)$ be corresponding solutions of problem (1)-(2) in the space $W_p^{(2,2)}(G)$. Then, the increment of functional (3) will be of the form

$$\Delta S(\upsilon)=\sum_{k=1}^N\left[\alpha_k\Delta u\left(\tau_k,\lambda_k^{(0)}\right)+\beta_k\Delta u\left(\mu_k^{(0)},\xi_k\right)+\gamma_k\Delta u\left(\mu_k^{(0)},\lambda_K^{(0)}\right)\right],\tag{4}$$

Obviously, here the function $\Delta u\in W_p^{(2,2)}(G)$ is a solution of the equation

$$(V_{2,2}\Delta u)(x)=\Delta\varphi(x),\tag{5}$$

satisfying trivial conditions

$$V_{i,j}^{(k)}\Delta u=0,\quad i=\overline{0,2},\ \ j=\overline{0,2},\ \ i+j<4,\tag{6}$$

where

$$\Delta\varphi(x)=\varphi(x,\upsilon(x)+\Delta\upsilon(x))-\varphi(x,\upsilon(x)).$$

The operator $V_{(k)}=\left(V_{0,0}^{(k)},V_{1,0}^{(k)},V_{0,1}^{(k)},V_{1,1}^{(k)},V_{2,0}^{(k)},V_{2,1}^{(k)},V_{0,2}^{(k)},V_{1,2}^{(k)},V_{2,2}\right)$ of problem (1)-(2) acts from $W_P^{(2,2)}(G)$ to $E_p^{(2,2)}\equiv R\times R\times R\times R\times L_p(G_1)\times L_p(G_1)\times L_p(G_2)\times L_p(G_2)\times L_p(G)$.

It is shown that the operator $V_{(k)}$ has the adjoimt operator $V_{(k)}^*=\left(\omega_{0,0}^{(k)},\omega_{1,0}^{(k)},\omega_{0,1}^{(k)},\omega_{1,1}^{(k)},\omega_{2,0}^{(k)},\omega_{2,1}^{(k)},\omega_{0,2}^{(k)},\omega_{1,2}^{(k)},\omega_{2,2}\right)$, that acts in the space $E_q^{(2,2)}\equiv R\times R\times R\times R\times L_q(G_1)\times L_q(G_1)\times L_q(G_2)\times L_q(G_2)\times L_q(G)$ and satisfies the condition

$$f\left(V_{(k)}u\right)=\iint_G f_{2,2}(x)(V_{2,2}u)(x)dG+\sum_{i=0}^1\sum_{j=0}^1f_{i,j}V_{i,j}^{(k)}u+$$

$$+\int_{G_1}\left[\sum_{j=0}^1f_{2,j}(x_1)\left(V_{2,j}^{(k)}u\right)(x_1)\right]dG_1+\int_{G_2}\left[\sum_{i=0}^1f_{i,2}(x_2)\left(V_{i,2}^{(k)}u\right)(x_2)\right]dG_2=$$

$$=\sum_{i=0}^1\sum_{j=0}^1\left(\omega_{i,j}^{(k)}f\right)D_1^iD_2^ju(\tau_k,\xi_k)+\sum_{j=0}^1\left(\omega_{2,j}^{(k)}f\right)(x_1)D_1^2D_2^ju(x_1,\xi_k)+$$

$$+\sum_{i=0}^1\left(\omega_{i,2}^{(k)}f\right)(x_2)D_1^iD_2^2u(\tau_k,x_2)+\iint_G(\omega_{2,2}f)(x)D_1^2D_2^2u(x)dG=\left(V_{(k)}^*f\right)(u),\tag{7}$$

where $f=\left(f_{0,0}, f_{1,0}, f_{0,1}, f_{1,1}, f_{2,0}(x_1), f_{2,1}(x_1), f_{0,2}(x_2), f_{1,2}(x_2), f_{2,2}(x)\right)$ is an arbitrary linear bounded functional in $E_p^{(2,2)}$ and u is an arbitrary function from $W_P^{(2,2)}(G)$ and $1/\mathrm{p}+1/\mathrm{q}=1$.

Now, in equality (7) instead of $u(x)$ we put the solution of problem (5)-(6), i.e. instead of function u we put the function Δu. Then, it is valid the equality

$$f\left(V_{(k)}\Delta u\right)=\iint_G f_{2,2}(x)\Delta\varphi(x)dG=\iint_G\left(\omega_{2,2}f\right)(x)D_1^2D_2^2\Delta u(x)dG=\left(V_{(k)}^* f\right)(\Delta u), \tag{8}$$

for all $f\in E_q^{(2,2)}$. In other words,

$$-\iint_G f_{2,2}(x)\Delta\varphi(x)dG+\iint_G\left(\omega_{2,2}f\right)D_1^2D_2^2\Delta u(x)dG=0, \tag{9}$$

The function $\Delta u(\mathrm{x})$ as an element of the space $W_p^{(2,2)}(G)$ satisfies trivial conditions (6). Using integral representation of functions from $W_p^{(2,2)}(G)$:

$$u(x)=\sum_{i=0}^{1}\sum_{j=0}^{1}(x_1-\tau_k)^i(x_2-\xi_k)^j D_1^iD_2^ju(\tau_k,\xi_k)+\sum_{j=0}^{1}(x_2-\xi_k)^j\int_{\tau_k}^{x_1}(x_1-\nu_1)D_1^2D_2^ju(\nu_1,\xi_k)d\nu_1+$$

$$+\sum_{\mathrm{i}=0}^{1}(x_1-\tau_k)^i\int_{\xi_k}^{x_2}(x_2-\nu_2)D_1^iD_2^2u(\tau_k,\nu_2)d\nu_2+\int_{\tau_k}^{x_1}\int_{\xi_k}^{x_2}(x_1-\nu_1)(x_2-\nu_2)D_1^2D_2^2u(\nu_1,\nu_2)d\nu_1d\nu_2$$

we get $\alpha_k\Delta u\left(\tau_k,\lambda_k^{(0)}\right)+\beta_k\Delta u\left(\mu_k^{(0)},\xi_k\right)+\gamma_k\Delta u\left(\mu_k^{(0)},\lambda_k^{(0)}\right)=\iint_G B_k(x)D_1^2D_2^2\Delta u(x)dG$, where

$$B_k(x)=\alpha_k(\tau_k-x_1)\left(\lambda_k^{(0)}-x_2\right)\theta(\tau_k-x_1)\theta\left(\lambda_k^{(0)}-x_2\right)+\beta_k\left(\mu_k^{(0)}-x_1\right)(\xi_k-x_2)\theta\left(\mu_k^{(0)}-x_1\right)\theta(\xi_k-x_2)+$$

$$+\gamma_k\left(\mu_k^{(0)}-x_1\right)\left(\lambda_k^{(0)}-x_2\right)\theta\left(\mu_k^{(0)}-x_1\right)\theta\left(\lambda_k^{(0)}-x_2\right);\ \theta(\tau)=\begin{cases}1, & \tau>0\\ 0, & \tau\le 0\end{cases}$$ is a Heaviside function;

Therefore, we can represent increment (4), of functional (3) in the form

$$\Delta S(\upsilon)=\iint_G B(x)D_1^2D_2^2\Delta u(x)dG, \tag{10}$$

where, $$B(x)=\sum_{k=1}^{N}B_k(x).$$

Now, using (9) we can write increment (10) in the form

$$\Delta S(\upsilon)=\iint_G\left[B(x)+\left(\omega_{2,2}f\right)(x)\right]D_1^2D_2^2\Delta u(x)dG-\iint_G f_{2,2}(x)\Delta\varphi(x)dG, \tag{11}$$

where

$$\left(\omega_{2,2}f\right)(x)\equiv\int_{x_1}^{h_1}\int_{x_2}^{h_2}(\nu_1-x_1)(\nu_2-x_2)a_{0,0}(\nu_1,\nu_2)f_{2,2}(\nu_1,\nu_2)d\nu_1d\nu_2+$$

$$+\int_{x_1}^{h_1}\int_{x_2}^{h2}(\nu_2-x_2)a_{1,0}(\nu_1,\nu_2)f_{2,2}(\nu_1,\nu_2)d\nu_1d\nu_2+\int_{x_1}^{h_1}\int_{x_2}^{h_2}(\nu_1-x_1)a_{0,1}(\nu_1,\nu_2)f_{2,2}(\nu_1,\nu_2)d\nu_1d\nu_2+$$

$$+\int_{x_1}^{h_1}\int_{x_2}^{h_2}a_{1,1}(\nu_1,\nu_2)f_{2,2}(\nu_1,\nu_2)d\nu_1d\nu_2+\int_{x_2}^{h_2}(\nu_2-x_2)a_{2,0}(x_1,\nu_2)f_{2,2}(x_1,\nu_2)d\nu_2+f_{2,2}(x)+$$

$$+\int_{x_2}^{h_2}a_{2,1}(x_1,\nu_2)f_{2,2}(x_1,\nu_2)d\nu_2+\int_{x_1}^{h_1}(\nu_1-x_1)a_{0,2}(\nu_1,x_2)f_{2,2}(\nu_1,x_2)d\nu_1+\int_{x_1}^{h_1}a_{1,2}(\nu_1,x_2)f_{2,2}(\nu_1,x_2)d\nu_1.$$

Since $\omega_{2,2}$ depends only on an element f, i.e. on $f_{2,2}$, the equality (11) is valid for all $f_{2,2}\in L_q(G)$. For simplifying expression (11) we introduce the equation

$$\left(\omega_{2,2}f_{2,2}\right)(x)+B(x)=0,\quad x\in G, \tag{12}$$

that will be said to be adjoint equation for optimal control problem (1)-(3) and as the function $f_{2,2}(x)$ we take the solution of equation (12) in $L_q(G)$. Then formula (11) will take the simple form: $\Delta S(\upsilon) = -\iint\limits_G f_{2,2}(x)\Delta\varphi(x)dG.$

Now, for the fixed $\rho = (\rho_1, \rho_2) \in G$ we consider the following needle shaped variation of admissible control $\upsilon(x)$: $\Delta\upsilon_\varepsilon(x) = \begin{cases} \hat{\upsilon} - \upsilon(x), & x \in G_\varepsilon \\ 0, & x \in G/G_\varepsilon, \end{cases}$ where $\hat{\upsilon} \in \Omega_\partial$, $\varepsilon > 0$ is a sufficiently small parameter, $G_\varepsilon = \left(\rho_1 - \frac{\varepsilon}{2}, \rho_1 + \frac{\varepsilon}{2}\right) \times \left(\rho_2 - \frac{\varepsilon}{2}, \rho_2 + \frac{\varepsilon}{2}\right).$

The control $\upsilon_\varepsilon(x)$, determined by the equality $\upsilon_\varepsilon(x) = \upsilon(x) + \Delta\upsilon_\varepsilon(x)$ is an admissible control for all sufficiently small $\varepsilon > 0$ and all $\hat{\upsilon} \in \Omega_\partial$, $\rho \in G$, called needle-shaped perturbation of the given control $\upsilon(x)$.

Obviously

$$\begin{aligned} S(\upsilon_\varepsilon) - S(\upsilon) &= -\iint\limits_G f_{2,2}(x)[\varphi(x,\upsilon(x) + \Delta\upsilon_\varepsilon(x)) - \varphi(x,\upsilon(x))]dG = \\ &= -\iint\limits_G f_{2,2}(x)[\varphi(x,\hat{\upsilon}) - \varphi(x,\upsilon(x))]dG, \end{aligned} \tag{13}$$

As the optimal control problem (1)-(3) is linear, the following theorem follows from (13).

Theorem. Let $f_{2,2}(x) \in L_q(G)$ be a solution of adjoint equation (12). Then for the optimality of the admissible control $\upsilon(x)$ it is necessary and sufficient that almost for all $x \in G$, the maximum condition be fulfilled

$$\max_{\hat{\upsilon} \in \Omega_\partial} H(x, f_{2,2}(x), \hat{\upsilon}) = H(x, f_{2,2}(x), \upsilon(x)),$$

where, $H(x, f_{2,2}, \upsilon) = f_{2,2}\varphi(x,\upsilon)$ is a Hamilton-Pontryagin function.

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