ON CONTROL PROBLEM IN SYSTEMS WITH CONCENTRATED PARAMETERS ON THE CLASS OF HEAVYSIDE FUNCTIONS

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A problem of optimal control by the objects described by ordinary differential equations on a class of control actions defined by Heavyside function is considered in the given article. A numerical method for solving the problem on determining the values of control actions and the moments of their switching on, based on obtained analytical formulas for the functional gradient, is proposed. The results of numerical experiments are given.

Particularly, let us consider the problem of minimization of the functional

$$J(u) = I(q,\theta) = \alpha_1 \int_0^T f_0(x,t)dt + \alpha_2 \Phi_1(x(T)) + \alpha_3 \Phi_2(q,\theta) \to \min_{(q,\theta)} \quad (1)$$

with conditions

$$\dot{x}_i(t) = f_i(x(t),t) + \sum_{j=1}^{L} b_{ij}(t) u_{ij}(t), \ 0 < t \le T, \ \ x(0) = x_0, \ i = \overline{1,n} \quad (2)$$

$$u_{ij}(t) = q_{ij}\chi(t-\theta_j), \ j = 1,...,L, \ i = \overline{1,n}. \quad (3)$$

Here $\chi(t-\theta_j) = \begin{cases} 0, & t < \theta_j, \\ 1, & t \ge \theta_j, \end{cases}$ is a Heavyside function, $x(t) = (x_1(t),...,x_n(t))$ phase vector; functions $f^0(x,t), f(x,t) = (f^1(x,t),..., f^n(x,t)), (b_1(t),...,b_L(t)),$ $\Phi_1(x), \Phi_2(q,\theta)$ of variables $(x,q,\theta,t) \in E^n \times E^L \times E^L \times [0,T]$ are given and are given and it is assumed to be continuous along with its derivatives on all the arguments. The control influences:

$u = (q,\theta), \ q = (q_1,...,q_L), \ \theta = (\theta_1,...,\theta_L)$

satisfy the following conditions:

$$\underline{q_i} \le q_{ij} \le \overline{q_i}, 0 \le \theta_i \le T, i = \overline{1,n}, \ j = \overline{1,L}. \quad (4)$$

Here $\underline{q_i}, \overline{q_i}, L, \alpha_1, \alpha_2$ and initial vector x_0 is given.

Control influences determined by vector-function $u = u(\bullet) \in E^{nL}$, where $u_{ij}(t) = u_{ij}(t; q_{ij}, \theta_j)$ each component of which is piecewise-constant function [1, 2] changing its values once time only, where $\theta = (\theta_1, \theta_2, ..., \theta_L)$ are the moments of the excitation of control influences q_{ij}.

The problem (1)–(3) form one side may be cconcerned to parametrical optimal control problems [3], form other side it is the problem of finite-dimensional optimization, to calculate of target functional (1) it is required to solve the problem Koshi and calculate the integral in (1).

With purpose of applying finite-dimensional optimization methods of the first order for the determining of optimal control, it will be obtained analytical formulas for the gradient of the functional of researching problem.

Let us consider the following Hamilton-Pontryagin function and adjoint system for the problem of (1)-(4):

$$H(\psi, x, u, t) = -f_0(x,t) + \sum_{i=1}^{n} \psi_i(t) f_i(x,t) + \sum_{i=1}^{n} \psi_i(t) \sum_{j=1}^{L} b_{ij}(t) q_{ij} \chi_{\theta_j}(t),$$

$$\dot{\psi}_i(t) = \alpha_1 \frac{\partial f_0(x(t),t)}{\partial x_i} - \sum_{k=1}^{n} \psi_k(t) \frac{\partial f_k(x(t),t)}{\partial x_i},$$

$$\psi_i(T) = -\alpha_2 \frac{\partial \Phi_1(x(T))}{\partial x_i(T)}, i = \overline{1,n},$$

where $\psi(t) \in E^n$. The increment of the functional (1) for admissible control $u(t)$ and its increment $\Delta u(t)$ can be written in the following way [4]:

$$\Delta I(q,\theta) = \Delta J(u) = J(u+\Delta u) - J(u) = -\alpha_1 \sum_{i=1}^{n} \sum_{j=1}^{L} \int_0^T \psi_i(t) b_{ij}(t) \Delta u dt +$$
$$+ \alpha_3 (\Phi_2(u+\Delta u) - \Phi_2(u)) + o(\|\Delta u\|).$$

For the first, to obtain the expression for $\frac{dJ(q,\theta)}{dq_{ij}}$ $i = \overline{1,n}$, $j = \overline{1,L}$, let us give increment Δq_{ij} on arqument q_{ij}, i.e. $\Delta u = (\Delta q, 0), \Delta q = (0,...,\Delta q_{ij},...,0)$. Let us write the functional increment in the following way:

$$\Delta_{q_{ij}} J(u) = -\alpha_1 \int_0^T \psi_i(t) b_{ij}(t) \Delta q_{ij} \chi(t-\theta_j) dt + \alpha_3 (\Phi_2(q+\Delta q,\theta) - \Phi_2(q,\theta)) + o(\|\Delta q_{ij}\|),$$
$$i = \overline{1,n},\ j = \overline{1,L}.$$

Dividing both parts by Δq_{ij} and according to $o(\|\Delta q\|)/\Delta q_{ij} \to 0$ and converging to limit at $\Delta q_{ij} \to 0$ we will get the next expression:

$$\frac{dJ(q,\theta)}{dq_{ij}} = -\alpha_1 \int_{\theta_j}^T \psi_i(t) b_{ij}(t) dt + \alpha_3 \frac{\partial \Phi_2(q,\theta)}{\partial q_{ij}},\ i = \overline{1,n},\ j = \overline{1,L}. \quad (5)$$

And now let us get the expression for $\frac{dJ(q,\theta)}{d\theta_j}$, $j = \overline{1,L}$ and that is why let us give increment $\Delta\theta_j$ on argument θ_j, at first assuming $\Delta\theta_j > 0$. Then Heavyside function will get the following increment:

$$\Delta\chi(t-\theta_j) = \chi(t-\theta_j+\Delta\theta_j) - \chi(t-\theta_j) = \begin{cases} 0, & t \notin [\theta_j, \theta_j + \Delta\theta_j], \\ -1, & t \in [\theta_j, \theta_j + \Delta\theta_j]. \end{cases}$$

Thus the increment of the functional obtaining at positive increment $\Delta\theta_j$ of argument θ_j, i.e. $\Delta u = (0, \Delta\theta), \Delta\theta = (0,...,\Delta\theta_j,....,0)$ will get the following form:

$$\Delta_{\theta_j} J(u) = -\alpha_1 \left(\sum_{i=1}^{n} \sum_{j=1}^{L} \int_0^T \psi_i(t) b_{ij}(t) q_{ij} \Delta\chi(t-\theta_j) dt \right) + \alpha_3 (\Phi_2(q,\theta+\Delta\theta) - \Phi_2(q,\theta)) + o(\|\Delta u\|) =$$
$$= \alpha_1 \left(\sum_{i=1}^{n} q_{ij} \int_{\theta_j}^{\theta_j+\Delta\theta_j} \psi_i(t) b_{ij}(t) dt \right) + \alpha_3 (\Phi_2(q,\theta+\Delta\theta) - \Phi_2(q,\theta)) + o(\|\Delta u\|).$$

At the case of negative increment of argument θ_j, i.e. at $\Delta\theta_j < 0$ the Heavyside function will get increment in the next form:

$$\Delta\chi(t-\theta_j) = \begin{cases} 0, & t \notin [\theta_j - |\Delta\theta_j|, \theta_j], \\ 1, & t \in [\theta_j - |\Delta\theta_j|, \theta_j]. \end{cases}$$

Then the functional increment can be written in the following way:

$$\Delta J_{\theta_j}(u) = -\alpha_1 \sum_{i=1}^{n} q_{ij} \left(\int_{\theta_j - |\Delta\theta_j|}^{\vartheta_j} \psi_i(t) b_{ij}(t) dt \right) + \alpha_3 (\Phi_2(q, \theta + \Delta\theta) - \Phi_2(q, \theta)) + o(\| \Delta u \|).$$

According to theorem on middle value [4] let us write the functional increment in the following form:

$$\Delta J_{\theta_j}(u) = \pm\alpha_1 \sum_{i=1}^{n} q_{ij} \psi_i(t) b_{ij}(t)|_{t=\theta_j} |\Delta\theta_j| + \alpha_3 (\Phi_2(q, \theta + \Delta\theta) - \Phi_2(q, \theta)) + o(\| \Delta u \|),$$

Where « + » corresponds with $\Delta\theta_j > 0$, but « – » with $\Delta\theta_j < 0$. Dividing both parts by $\Delta\theta_j$ and converging to limit at $\Delta\theta_j \to 0$, independent on sing of $\Delta\theta_j$ we will get the next expression:

$$\frac{dI(q,\theta)}{d\theta_j} = \alpha_1 \sum_{i=1}^{n} q_{ij} \psi_i(t) b_{ij}(t)|_{t=\theta_j} + \alpha_3 \frac{\partial \Phi_2(q,\theta)}{\partial \theta_j}, \quad j = \overline{1, L}. \tag{6}$$

Thus, the next theorem is proved.

Theorem: The components of the functional gradient on parameters of the control influences $(q,\theta) \in E^{(n+1)L}$ in the problem (1), (4) are determining by the formulas (5), (6).

Remark: If **a**n optimal control problem by the objects described by system of non-linear ordinary differential equations

$$\dot{x}_i(t) = f_i(x(t), u_1(t), t), t \in (0, T],$$
$$x_i(0) = x_{0i}, i = \overline{1, n},$$

where the components of control influnce $u_1(t) \in E^{r_1}$ are from the class of Heavyside functions, then it can be obtained formulas for the components of functional gradient of control actions and the moments of their switching on in the following form, analogically to (5), (6) :

$$\frac{dI(q,\theta)}{dq_{ij}} = -\int_{\theta_j}^{T} \frac{\partial f_i(x(t), u_1(t), t)}{\partial q_{ij}} \psi_i(t) dt + \alpha_3 \frac{\partial \Phi_2(q,\theta)}{\partial q_{ij}}, \; i = \overline{1, n}, \; j = \overline{1, L},$$

$$\frac{dI(q,\theta)}{d\theta_j} = \sum_{i=1}^{n} q_{ij} [\frac{\partial f_i(x(t), u_1(t), t)}{\partial q_{ij}} \psi_i(t)]_{t=\theta_j} + \alpha_3 \frac{\partial \Phi_2(q,\theta)}{\partial \theta_j}, \quad j = \overline{1, L}.$$

Where $\psi(t) \in E^n$ is a solution of the next adjoint system:

$$\dot{\psi}_i(t) = \alpha_1 \frac{\partial f_0(x(t), t)}{\partial x_i} - \sum_{k=1}^{n} \psi_k(t) \frac{\partial f_k(x(t), u_1(t), t)}{\partial x_i},$$

$$\psi_i(T) = -\alpha_2 \frac{\partial \Phi_1(x(T))}{\partial x_i(T)}, i = \overline{1, n}.$$

By using the receiving formulas for functional gradient, let us yield the results of applying them in the next test problem.

$$\dot{x}(t) = tx(t) + (t+2) \sum_{i=1}^{2} q_i \chi(\theta_i), \;\; 0 < t \le 1, \;\; x(0) = 1,$$

$$0 \le \theta_i \le 1, 0 \le q_i \le 25, \; i = 1,2,$$

$$J(u) = I(q,\theta) = \int_0^1 x^2(t) dt + 0.2\left((q_1 - 5)^2 + (q_2 - 7)^2\right) + 5\left((\theta - 0.5)^2 + (\theta - 0.8)^2\right).$$

The accurate optimal control for this problem is unknown.

The problem was numerically solved by using formulas (5), (6) with step $h = 0.002$. The results of numerical experiments by using the method of the projection of adjoint gradients for different initial values of control vector with optimization accuracy $\varepsilon = 0{,}001$ are shown on the table 1.

Table 1

The numerical results of the problem

№	(q^0,θ^0)	(q^*,θ^*)	J^0	J^*	The number of iter.
1	(12;18) (0,4;0,2)	(4,9278;6,9063) (0,9513;0,9880)	1761,5093	3,6959	5
2	(6;14) (0,2;0,8)	(4,9078;7,1254) (0,9491;0,9880)	173,9044	3,6983	21
3	(3;6) (0,1;0,4)	(4,9876;7,0573) (0,9520;0,9880)	169,8865	3,6954	6
4	(8,2;20,1) (0,71;0,84)	(4,7143;6,8455) (0,9510;0,9861)	72,6134	3,6990	15

The constructive analytical formulas for the gradient of the target functional of the considered problem are obtained which allow using first order optimization methods for solving the optimal control problem.

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Reference

1. Ayda-zade K.R., Rahimov A.B. The numerical solution of optimal control on the class of piecewise constant functions (In Russian) // Izvestiya NAN Azerbaijan, ser. FTMN, 2005, v. XXV, No.2, pp. 99-104.
2. Moiseev A.A. Optimal control on discret control influences (In Russian) //Automation and telemechanics, 1991, No.9, pp. 123-132.
3. Pontryagin L.S., Boltyanskiy V.Q., Qamkrelidze R.V., Mishenko E.F. The mathematical theory of optimal processes. (In Russian) M.: Fizmatgiz, 1969, 384 p.
4. Qabasov R., Kirillova F.M., The maximum principle in the theory of optimal control (In Russian) M., Nauka and technika, 1974, 272 p.
5. Fichtengols Q.M. The course of differential and integral, (In Russian) V. II, Publication «Nauka», Moscow, 1970, 800 p.