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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

DEVELOPMENT OF MATHEMATICAL MODELS OF QUEUING SYSTEMS WITH MULTIPLE INVENTORY REPLENISHMENT SOURCES

Specialty: 3338.01 – Systems analysis, control, and information processing (modeling and management)

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GENERAL CHARACTERISTICS OF THE WORK

Relevance of the Topic. Queuing-Inventory Systems (QIS) are systems that arise from combining, within a unified framework, the essential characteristics of queuing systems and inventory control systems. In classical queuing systems, arriving requests (customers), after receiving service, do not remove any system resources such as servers, waiting places, or other service facilities. In inventory control systems, on the other hand, there are no service stations (i.e., the service time is assumed to be zero), and therefore the formation of customer queues is impossible.

However, in many real-life service systems, such as logistics networks, supply chains for material resources, fuel stations, online stores, blood banks, retail trade systems, and similar service environments, the fulfillment of incoming requests requires both an available server and a certain amount of system resources (inventory). In such systems, queues arise due to the random nature of customer arrivals, service durations, and replenishment lead times.

The construction of adequate mathematical models for QIS requires the combined application of the theories of queuing systems and inventory control systems. For this reason, during the last three decades, QIS have been intensively investigated by numerous researchers in different countries around the world.

An analysis of the existing literature shows that the majority of studies are devoted to QIS models in which inventories are replenished from only a single source. In practice, however, unexpected natural disasters and public health emergencies (such as earthquakes, fires, wars, terrorist attacks, epidemics, and similar events), as well as situations in which a supplier sharply increases inventory prices, may make it impossible or economically infeasible to obtain supplies from a specific source. Therefore, in order to increase the reliability and resilience of real QIS and to reduce operational risks, it is essential to use multiple supply sources.

Despite its theoretical and practical significance, multi-source QIS have not been sufficiently studied in the existing literature. In the present dissertation, mathematical models of queuing-inventory systems with replenishment from two supply sources are proposed,

and exact as well as approximate methods for their analysis are developed.

For these reasons, the topic of the dissertation is scientifically relevant and of considerable practical importance.

Object and subject of the research. The object of the research is a single-server queuing-inventory system replenished from two sources.

The subject of the research is the mathematical modeling and analytical investigation of the operating mechanisms of such systems, including Poisson flows of customers demanding inventory and customers destroying inventory, the service process, and the behavior of impatient customers.

Within the research framework, the state space of the system is constructed based on a Markov process, transition intensities are determined, stationary probability distributions are obtained, and key performance measures and the expected total cost function are evaluated using analytical and computational methods.

Aim and objectives of the research. The main objective of the dissertation is to develop mathematical models that adequately describe the operation of queuing-inventory systems with replenishment from two different sources.

To achieve this objective, the following problems have been solved:

- Replenishment policies for queuing-inventory systems supplied from two different sources have been proposed;
- Efficient methods have been developed for calculating system characteristics and minimizing the expected total economic costs;
- Ergodicity conditions for queuing-inventory systems with infinite buffer capacity and two supply sources have been derived.

Research methods. To solve the stated problems, methods of applied probability theory, operations research, Markov chains (including quasi-birth-and-death processes), matrix theory, the space-merging (state aggregation) method, the matrix-geometric method, and computational techniques have been employed.

Main provisions submitted for defense. The following results are submitted for defense:

- Replenishment policies for QIS supplied from fast and slow sources;
- Development of the space-merging method for calculating the stationary distribution of QIS with finite but large buffers;
- Development of the matrix-geometric method for calculating the stationary distribution of QIS with infinite buffers;
- Calculation of performance measures and solution of the problem of minimizing the expected total cost.

Scientific novelty of the research. The main scientific contributions of the dissertation are as follows:

1. Construction of two-source QIS models. A unified approach has been proposed for modeling QIS supplied from two sources with different characteristics (fast/expensive and slow/cheap).

2. Development of new replenishment policies. New replenishment policies based on classical (s, Q), (s, S), and $(S-1, S)$ policies have been defined for QIS supplied from two heterogeneous sources. In addition, a completely new hybrid replenishment policy has been proposed.

3. Application of efficient methods for finding stationary probabilities. For **finite-buffer** systems of large size, the space-merging method has been applied to obtain simple and highly accurate approximate formulas for stationary probabilities.

For **infinite-buffer systems**, the matrix-geometric method has been applied; ergodicity conditions necessary for system stability (existence of a stationary regime) have been derived, and algorithms for computing stationary probabilities have been developed.

4. Analytical determination of performance measures. For each proposed replenishment policy, explicit formulas have been obtained for key performance indicators such as the average queue length, average inventory level, probability of order loss, etc.

5. Minimization of the Expected Total Cost function. To evaluate economic efficiency, the Expected Total Cost (ETC) function has been constructed. Based on this function, optimization problems

for determining control parameter values that minimize the ETC have been solved.

Theoretical and practical significance. The theoretical significance lies in extending QIS theory to the case of complex supply chains with two supply sources.

The practical significance lies in the applicability of the developed models and algorithms to real logistics systems, warehouse management, manufacturing enterprises, and retail networks. These models enable managers to determine optimal inventory levels, reorder points, and order quantities from different sources, thereby minimizing costs and increasing customer satisfaction.

Approbation and implementation. The main results of the dissertation were presented and discussed at the following international and national scientific conferences:

1. 25th International Scientific Conference Distributed Computer and Communication Networks (DCCN2022), Moscow, September 26-30, 2022;

2. "February Reports 2022", VII International Scientific-Practical Youth Conference "Creative Potential of Youth in Solving Aerospace Problems", National Aviation Academy, Baku, February 2-4, 2022;

3. II Republican Scientific Conference "Fundamental Problems of Mathematics and Application of Intelligent Technologies in Education", Sumgait State University, December 15-16, 2022;

4. "February Reports 2023", VIII International Scientific-Practical Youth Conference "Creative Potential of Youth in Solving Aerospace Problems", National Aviation Academy, Baku, February 1-3, 2023;

5. Management and Control in Solving Engineering Problems (MaCoSEP-2025), Baku Engineering University, March 12-14, 2025.

The dissertation was carried out at Baku Engineering University.

Structure and volume of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion, and a bibliography containing 118 references, as well as 68 tables and 13 figures. The total length of the work is 148 pages.

Publications. A total of 12 scientific works have been published on the topic of the dissertation. Among them, 7 articles were published in peer-reviewed journals. Four of these articles were published in journals indexed in both Web of Science and Scopus (Q1, Q3, and Q4 categories). The author's h-index in the Scopus database is 3, with a total of 43 citations.

MAIN CONTENT OF THE DISSERTATION

In the **Introduction**, the relevance of the topic is substantiated, and the purpose of the research, its scientific novelty, and practical significance are explained.

In the **first chapter** of the dissertation, the theoretical foundations, historical development, main components, performance measures, and the current state of modern mathematical models of QIS are systematically analyzed.

In this chapter, the formation of the QIS concept in the scientific literature, the initial fundamental works, and the systematic studies conducted in recent decades are summarized. It is particularly emphasized that QIS theory has developed rapidly over the past 30 years, has gained numerous application areas, and has become the subject of special issues in international scientific journals. This confirms both the theoretical and practical significance of the field.

This chapter also provides a description of the main structural components of the studied QIS models. Classical replenishment policies are explained, and their implementation mechanisms and characteristics are presented comparatively. At the same time, it is emphasized that the existence of several supply sources (for example, slow-cheap and fast-expensive sources) significantly affects the economic and operational performance indicators of the system, and in such cases, the development of new, more flexible replenishment policies is necessary.

The main performance measures of QIS are divided into two groups:

1. Measures important from the customers' point of view (the average number of customers in the system and in the queue, waiting time, loss probability, etc.);

2. Measures important for system managers (the average inventory level, order intensity, the volume of orders arriving from different sources, the inventory destruction intensity, etc.).

It is shown that in order to calculate these measures, it is necessary to determine the joint stationary probability distribution of the system. This requires the construction of two-dimensional Markov chains and the application of the matrix-geometric method and other numerical methods (for example, the space merging method).

In the final part of the chapter, the current state of QIS models is classified based on an extensive literature review. A comprehensive analysis of the existing literature shows that the majority of research in QIS theory focuses on cases where inventories are replenished from only one source (supplier) or from sources with identical characteristics. Although studies on two-source QIS models have been conducted in recent years, these studies do not fully address the internal constraints of the system and the physical adaptation of replenishment policies.

For the models studied in the dissertation, the following assumptions are adopted:

- The maximum capacity of the warehouse of the single-server system is S ($S < \infty$), and c-customers arrive at the system according to a Poisson process with intensity λ;
- An arriving c-customer is immediately admitted to service when the server is idle and the inventory level is positive; if the inventory level is positive and the server is busy, then the c-customer joins a finite queue. The maximum buffer size for waiting c-customers is taken as $N - 1$ ($1 < N < \infty$); if the buffer is full at the moment of arrival, then the c-customer is lost with probability one (leaves the system without receiving a product);
- If at the moment of arrival the inventory level is zero but there is free space in the buffer, then the c-customer either joins the queue with probability φ_1 or leaves the system with probability φ_2, where $\varphi_1 + \varphi_2 = 1$;
- After the completion of service, each c-customer either receives no product with probability σ_1 or receives a product with probability σ_2, where $\sigma_1 + \sigma_2 = 1$. In both cases, the service time of

c-customers follows an exponential distribution, but with different mean values: if the *c*-customer does not receive a product, the mean service time equals μ_1^{-1}; otherwise, it equals μ_2^{-1};

- A *c*-customer at the end (or at any position) of the queue becomes impatient when the inventory level drops to zero; that is, in such cases, a *c*-customer in the queue waits for a random time. It is assumed that this time is exponentially distributed with mean τ^{-1}, and after this time expires, the customer leaves the system without receiving a product;
- In addition to *c*-customers, destructive customers (*d*-customers) arrive at the system according to a Poisson process with intensity κ. When a *d*-customer arrives, the inventory level immediately decreases by one unit. It is assumed that a *d*-customer destroys the inventory in the warehouse; even if the warehouse is empty, the *d*-customer may destroy the product allocated for the *c*-customer being served. If the inventory level is zero, then an arriving *d*-customer does not affect the operation of the system;
- Inventory replenishment can be carried out either by a slow supplier (SS) or by a fast supplier (FS). The delivery time from each source follows an exponential distribution; that is, if an order is placed with SS, the mean waiting time for delivery equals ν_1^{-1}, and if an order is placed with FS, this time equals ν_2^{-1}, where $\nu_2 > \nu_1$.

In the **second chapter**, mathematical models of queueing-inventory systems with replenishment from two different sources and finite buffer capacity are developed and studied. Four different replenishment policies are considered:

- (s, Q) – If the inventory level falls to the reorder point s, where $0 < s < S/2$, then an order is placed with the S. If the inventory level falls to the critical level r $(0 \le r < s)$, then the order previously placed with the S is immediately canceled and an order is placed with the FS. The size of each order is fixed as $Q = S - s$.
- (s, S) – If the inventory level falls to the point s, where $0 < s < S$, then an order is placed with the S. If the inventory level falls to the critical level r $(0 \le r < s)$, then the order previously placed with the S is immediately canceled and an order is placed with

the FS. The order size is taken as $Q = S - m$, where m is the current inventory level.

• $(S-1, S)$ – Each time one unit of product is sold, a new order is placed immediately.

• Hybrid – If the system inventory level falls to s, where $0 < s < S/2$, an order is placed with the S according to the (s, Q) policy. If the inventory level reaches the critical threshold r, where $0 \le r < s$, then the previous order placed with the S is immediately canceled and a new order is placed with the FS according to the (s, S) policy.

For the sake of specificity, only the (s, Q) replenishment policy is explained here.

The mathematical model of the studied system is a two-dimensional Markov chain (2DMC). The states of this chain are described in the form (n, m), where n denotes the total number of c-customers in the system (in the buffer or in service), $n = 0, 1, \dots, N$, and m denotes the current inventory level, $m = 0,1, \dots, S$. The state space of the considered 2DMC is defined as:

$$E = \{0,1, \dots, N\} \times \{0,1, \dots, S\}. \quad (1)$$

The transition intensity from state $(n_1, m_1) \in E$ to state $(n_2, m_2) \in E$ is denoted by $q\big((n_1, m_1); (n_2, m_2)\big)$, and its generator is denoted by Q:

$$Q = \big(q\big((n_1, m_1); (n_2, m_2)\big): (n_1, m_1), (n_2, m_2) \in E\big). \quad (2)$$

Taking into account the operating mechanism of the system, the elements of the generator matrix are obtained as follows:

$$q\big((n, m); (n+1, m)\big) = \lambda\varphi_1 I(n < N) \cdot I(m = 0); \quad (3)$$

$$q\big((n, m); (n+1, m)\big) = \lambda I(n < N) \cdot I(m > 0); \quad (4)$$

$$q\big((n, m); (n-1, m)\big) = \mu_1\sigma_1 \cdot I(m > 0); \quad (5)$$

$$q\big((n, m); (n-1, m-1)\big) = \mu_2\sigma_2 \cdot I(m > 0); \quad (6)$$

$$q\big((n, m); (n, m-1)\big) = \kappa \cdot I(m > 0); \quad (7)$$

$$q\big((n, 0); (n-1, 0)\big) = \tau \cdot I(n > 0); \quad (8)$$

$$q\big((n, m); (n, m+S-s)\big) = \nu_1 \cdot I(r < m \le s); \quad (9)$$

$$q\big((n, m); (n, m+S-s)\big) = \nu_2 \cdot I(0 \le m \le r). \quad (10)$$

Here, $I(A)$ denotes the indicator function of event A: if event A occurs, then $I(A) = 1$; otherwise, $I(A) = 0$.

The steady-state probabilities of the constructed 2DMC are obtained as a result of solving the system of global balance equations (SGBE). These equations form a system of linear algebraic equations of dimension $(N + 1)(S + 1)$. For relatively small values of the system parameters, this SGBE can be solved using available software. However, for large-scale SGBE computational difficulties arise. To overcome these difficulties, the space-merging method (SMM), developed by A.Z. Melikov, Corresponding Member of ANAS and Doctor of Technical Sciences and Professor, for 2DMCs is used [1].

For the applicability of SMM in the considered model, the following assumption is adopted: $\lambda \gg max(\kappa, \nu)$. Under these conditions, the initial state space (1) of the studied 2DMC is partitioned as follows:

$$E = \bigcup_{m=0}^{S} E_m \ , E_{m_1} \cap E_{m_2} = \emptyset \ , \quad m_1 \neq m_2, \qquad (11)$$

where

$$E_m = \{(n, m) \in E : n = 0,1, \dots, N\}, m = 0,1, \dots, S. \qquad (12)$$

The subset E_m indicated in equality (11) consists of those states of the original state space in which the inventory level is m, regardless of the number of c-customers.

According to SMM, each subset E_m is combined into a single state $\langle m \rangle$, and the aggregated state space is expressed as:

$$\hat{E} = \{\langle m \rangle : m = 0,1, \dots, S\}.$$

In the original state space, the aggregation function $U: E \to \hat{E}$ is defined as:

$$U((n, m)) = \langle m \rangle \ , (n, m) \in E.$$

According to the proposed SMM approach, first the problem of finding the stationary probabilities of the states within each subset $E_m, m = 0, 1, \dots, S$, in partition (11) is considered. In each subset E_m, the stationary distributions coincide with the stationary distribution of

[1] Ponomarenko, L. Performance Analysis and Optimization of Multi-Traffic on Communication Networks. / L.Ponomarenko, C.S.Kim, A.Melikov. -London: Springer, – 2010. – 194 p.

the M/M/1/N system. In these decomposed models, the stationary probabilities of states $(n,m) \in E_m$, denoted by $\rho_m(n)$, are calculated as follows:

$$\rho_m(n) = \begin{cases} \theta^n \dfrac{1-\theta}{1-\theta^{N+1}}, & m > 0, n = 0,1,\dots,N, \\ \psi^n \dfrac{1-\psi}{1-\psi^{N+1}}, & m = 0, n = 0,1,\dots,N. \end{cases} \quad (13)$$

where $\theta = \lambda/\mu_1\sigma_1$ and $\psi = \lambda\varphi_1/\tau$.

The transition intensity from the aggregated state $\langle m_1 \rangle \in \hat{E}$ to another aggregated state $\langle m_2 \rangle \in \hat{E}$ is denoted by $q(\langle m_1 \rangle, \langle m_2 \rangle)$. These quantities are calculated according to the following formula:

$$q(\langle m_1 \rangle, \langle m_2 \rangle) = \begin{cases} \nu_2, & 0 \le m_1 \le r, m_2 = m_1 + S - s, \\ \nu_1, & r+1 \le m_1 \le s, m_2 = m_1 + S - s, \\ \omega, & m_1 > 0, m_2 = m_1 - 1, \end{cases} \quad (14)$$

where

$$\omega = \mu_2\sigma_2\big(1 - \rho(0)\big) + \kappa. \quad (15)$$

Using formulas (14), the balance equations for finding the stationary probabilities $\pi(\langle m \rangle)$, $\langle m \rangle \in \hat{E}$, of the one-dimensional Markov chain describing the aggregated model are constructed, and it is shown that these probabilities are obtained as follows:

$$\pi(\langle m \rangle) = \begin{cases} \alpha_m \pi(\langle 0 \rangle), & 1 \le m \le r, \\ \beta_m \pi(\langle 0 \rangle), & r+1 \le m \le s, \\ \beta_s \pi(\langle 0 \rangle), & s+1 \le m \le S-s, \\ \eta_m \pi(\langle 0 \rangle), & S-s+1 \le m \le S, \end{cases} \quad (16)$$

$$\pi(\langle 0 \rangle) = \left(1 + \sum_{m=1}^{r} \alpha_m + \sum_{m=r+1}^{s} \beta_m + (S-2s)\beta_s + \sum_{m=S-s+1}^{S} \eta_m\right)^{-1}. \quad (17)$$

Here the following notations are used:

$$\eta_m = \frac{1}{\omega} \sum_{i=m-S+s}^{s} \chi_i x_i, S - s + 1 \le m \le S; \quad (18)$$

$$\chi_i = \begin{cases} \alpha_i, & 1 \le i \le r, \\ \beta_i, & r+1 \le i \le s; \end{cases} \tag{19}$$

$$x_i = \begin{cases} \nu_2, & 1 \le i \le r, \\ \nu_1, & r+1 \le i \le s. \end{cases} \tag{20}$$

$$\alpha_m = a_2(1+a_2)^{m-1}, \tag{21}$$

$$\beta_m = a_2(1+a_2)^r(1+a_1)^{m-r-1}; \tag{22}$$

$$a_k = \frac{\nu_k}{\omega}, \quad k = 1, 2. \tag{23}$$

Finally, according to SMM, using relations (13) and (16), the approximate values of the stationary probabilities of the 2DMC constructed as the mathematical model of the studied QIS are calculated as follows:

$$\hat{p}(n,m) = \rho_m(n) \cdot \pi(\langle m \rangle), (n,m) \in E, \tag{24}$$

The average number of customers in the system (L_{av}) is found as the mathematical expectation of the corresponding random variable:

$$L_{av} = \sum_{n=1}^{N} n \sum_{m=0}^{S} p(n,m) \ . \tag{25}$$

To calculate the loss probability of consumer customers (*c*-customers), denoted by (*PL*), the following formula is obtained:

$$PL = \varphi_2 \sum_{n=0}^{N} p(n,0) \ + \sum_{m=0}^{S} p(N,m) + \\ + \frac{\tau}{\tau + \lambda\varphi_1 + \nu_2} \sum_{n=1}^{N-1} p(n,0) + \frac{\tau}{\tau + \nu_2} p(N,0). \tag{26}$$

The average inventory level in the warehouse (S_{av}) is found as the mathematical expectation of the corresponding random variable:

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{N} p(n,m) \ . \tag{27}$$

The average volume of orders delivered from the regular source $V_{av}(1)$ and from the urgent source $V_{av}(2)$ is also found as the mathematical expectation of the corresponding random variables:

$$V_{av}(1) = \sum_{m=r+1}^{S} (S-s) \sum_{n=0}^{N} p(n,m)\,; \tag{28}$$

$$V_{av}(2) = \sum_{m=0}^{r} (S-s) \sum_{n=0}^{N} p(n,m)\ . \tag{29}$$

The average destruction rate of the system inventory (*DRS*) is calculated by:

$$DRS = \kappa \left(1 - \sum_{n=0}^{N} p(n,0)\right). \tag{30}$$

The average intensities of orders delivered from SS and FS are calculated as follows:

$$RR_1 = \kappa p(0, s+1) + (\mu_2\sigma_2 + \kappa) \sum_{n=1}^{N} p(n, s+1). \tag{31}$$

$$RR_2 = \kappa p(0, r+1) + (\mu_2\sigma_2 + \kappa) \sum_{n=1}^{N} p(n, r+1)\ . \tag{32}$$

This approach is convenient for small and medium-sized models. To calculate approximate values of the performance measures for large-scale models, the following formulas are obtained:

$$S_{av} \approx \sum_{m=1}^{S} m\pi(\langle m \rangle)\ ; \tag{33}$$

$$V_{av}(1) \approx (S-s) \sum_{m=r+1}^{s} \pi(\langle m \rangle)\,; \tag{34}$$

$$V_{av}(2) \approx (S-s) \sum_{m=0}^{r} \pi(\langle m \rangle)\,; \tag{35}$$

$$L_{av} \approx \sum_{n=1}^{N} n\left(\big(1 - \pi(\langle 0 \rangle)\big)\rho(n) + \pi(\langle 0 \rangle)\rho_0(n)\right); \tag{36}$$

$$DRS \approx \kappa\big(1 - \pi(\langle 0 \rangle)\big)\ ; \tag{37}$$

$$RR_1 \approx \pi(\langle s+1 \rangle)\left(\kappa\rho(0) + (\mu_2\sigma_2 + \kappa)\big(1 - \rho(0)\big)\right); \tag{38}$$

$$RR_2 \approx \pi(\langle r+1\rangle)\left(\kappa\rho(0) + (\mu_2\sigma_2 + \kappa)\big(1-\rho(0)\big)\right); \quad (39)$$

$$PL \approx \varphi_2\pi(0)\big(1-\rho_0(N)\big) + \\ + \frac{\tau}{\tau+\lambda\varphi_1+\nu_2}\pi(0)\big(1-\rho_0(0)-\rho_0(N)\big) + \\ + \frac{\tau}{\tau+\nu_2}\rho_0(N)\pi(0)\big(1-\pi(0)\big)\rho(N) + \pi(0)\rho_0(N)\,. \quad (40)$$

Using the formulas obtained in the dissertation for the performance measures, their dependence on the initial system parameters was investigated. For this purpose, in the hypothetical model the following fixed values were assigned to the initial system parameters:

$$\lambda = 8,\ \kappa = 2,\ \mu_1 = 45,\ \mu_2 = 15,\ \tau = 1.5,\ \nu_1 = 2,\ \nu_2 = 8,$$
$$\sigma_1 = 0.4,\ \varphi_1 = 0.6, r = 5,\ S = 30,\ N = 100.$$

Considering the limited length of the abstract, only the dependence of the performance measures on the intensity of *c*-customers is presented in the following table.

Table 1. Dependence of performance measures on parameter λ

λ	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	*DRS*	RR_1	RR_2	*PL*
8	3.29	0.08	5.63	0.80	2	0.08	2.45×10^{-2}	4.22×10^{-5}
9	3.40	0.10	5.78	1.01	2	0.09	2.94×10^{-2}	6.48×10^{-5}
10	3.50	0.11	5.92	1.26	2	0.10	3.46×10^{-2}	9.50×10^{-5}
11	3.60	0.13	6.06	1.58	2	0.10	4.06×10^{-2}	1.34×10^{-4}
12	3.69	0.15	6.18	2.02	2	0.11	4.60×10^{-2}	1.83×10^{-4}

In this chapter, the considered model was also analyzed under the (s,S), $(S-1,S)$, and hybrid replenishment policies. Due to the limited size of the abstract, these results are not presented here.

One of the most important issues in these models is the minimization of the total costs generated by their operation. In real-life systems, the load parameters (i.e., the arrival intensities of orders and their service times), as well as the intensities of destructive orders, are typically uncontrollable parameters. In this case, the controllable parameters become those determining the inventory levels at which regular and emergency orders should be placed. In other words, by appropriately selecting the parameters s and r in the considered replenishment policies, the total system cost can be minimized.

For the considered QIS models, the Expected Total Cost (ETC) function is defined as

$$\mathrm{ETC}(s,r) = \sum_{i=1}^{2}\big(K_i + c_r(i)V_{av}(i)\big)RR_i + c_cRR_2 + c_hS_{av} + \\ +c_dDRS + c_l\lambda PL + c_wL_{av}\,, \qquad (41)$$

where:

- K_i – fixed cost of placing an order to Source-i ($i = 1,2$),
- $c_r(i)$ – transportation cost per unit of inventory from Source-i ($i = 1,2$),
- c_c – penalty cost for canceling an order placed to Source-1,
- c_h – holding cost per unit of inventory,
- c_d – destruction cost per unit of inventory,
- c_l – loss cost of one *c*-customer,
- c_l – waiting cost of one *c*-customer in the queue.

Source-1 refers to the slow source (SS), while Source-2 refers to the fast source (FS).

The optimization problem is formulated as follows: determine the optimal values (s^*, r^*) of the parameters s and r in the replenishment strategy such that function (41) attains its minimum value. That is,

$$(s^*, r^*) = arg \min_{(s,r)\in X} TEC(s,r)\,, \qquad (42)$$

where X is the set of all feasible pairs (s,r) satisfying $0 \le r < s < S/2$.

Problem (41)-(42) always has a solution since the feasible set is discrete and finite. Using appropriate software tools, the problem can be solved easily. In all numerical experiments, the following parameter values were used for comparison:

$$S = 22, \lambda = 8, \kappa = 2, \mu_1 = 45, \mu_2 = 15, \tau = 1.5, \nu_1 = 2, \nu_2 = 8,$$
$$K_1 = 100, K_2 = 200,\ c_r(1) = 50, c_r(2) = 100, c_c = 50, c_h = 35,$$
$$c_d = 75, c_l = 200, c_w = 50, N = 100\,.$$

All cost parameters in the model are expressed in conventional monetary units, and their units of measurement are determined in accordance with the structure of the cost function.

The following results were obtained.

Table 2. Results of formula (41) for the (s, Q) replenishment policy

r	s								
	10	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**
1	870.62	850.90	835.46	824.94	***820.45***	823.69	837.22	864.91	912.74
2	872.35	853.23	838.68	829.49	827.00	833.34	851.80	887.57	
3	879.47	862.48	850.85	845.75	849.10	863.98	895.31		
4	892.02	878.78	872.33	874.49	888.29	918.65			
5	910.98	903.58	905.28	919.10	950.08				
6	938.43	939.84	954.14	986.50					
7	977.88	992.76	1026.87						
8	1035.16	1071.21							
9	1120.16								

Table 3. Results of formula (41) for the (s, S) replenishment policy

r	s								
	10	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**
1	896.90	885.21	875.95	869.80	***867.73***	871.11	881.95	903.24	939.62
2	896.22	884.54	875.36	869.44	867.88	872.30	885.20	910.49	
3	899.59	889.07	881.50	877.89	879.71	889.26	910.23		
4	906.67	898.44	894.07	895.01	903.51	923.23			
5	917.85	913.31	914.16	922.68	942.6				
6	934.27	935.39	944.45	965.31					
7	958.00	967.89	990.1						
8	992.67	1016.55							
9	1044.60								

Table 4. Results of formula (41) for the hybrid replenishment policy

r	s								
	10	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**
1	794.53	775.68	759.29	745.44	734.26	725.93	720.63	718.50	719.60
2	791.67	772.39	755.32	740.39	727.51	716.50	706.98	***698.15***	
3	791.46	772.31	755.19	739.84	725.90	712.72	699.06		
4	792.97	774.24	757.28	741.59	726.36	710.09			
5	795.64	777.34	760.35	743.74	725.81				
6	798.98	780.80	763.02	743.79					
7	802.37	783.44	762.97						
8	804.69	782.92							
9	803.51								

The minimum ETC values in the tables are marked in bold. In the second chapter, the efficiency of different replenishment policies was analyzed in detail.

Although performance measures characterize individual aspects of system operation – such as service level, demand satisfaction rate, and resource utilization efficiency – they do not provide a fully objective assessment from an overall economic perspective.

Indeed, the same performance measure may correspond to different financial outcomes under different replenishment policies. For example, even if the same average inventory level is observed under the (s, Q) and (s, S) policies, one policy may involve small and frequent orders while the other involves larger and less frequent orders, leading to significant differences in ordering and delivery costs. Moreover, in some cases, even if system performance improves (e.g., reduced waiting time), holding or emergency delivery costs may increase, resulting in a higher total cost.

For this reason, comparing different replenishment policies in terms of the ETC function provides a more accurate and objective evaluation of their real effectiveness. The ETC integrates all relevant costs – holding, ordering, service, delay, emergency delivery, and cancellation costs – into a single measure, thereby fully reflecting the overall economic efficiency of the system.

In the **third chapter**, mathematical models of QIS with two supply sources and an infinite buffer are developed and analyzed. As in the second chapter, four different replenishment policies are considered: (s, Q), (s, S), $(S-1, S)$, and the hybrid policy.

For specificity, here we consider only the QIS model that uses the (s, S) replenishment policy.

In the considered model, the assumptions given in the second chapter remain valid. The only difference lies in the infinite buffer capacity and in the replenishment mechanism.

The state space is defined as

$$E = \{(n, m): n \geq 0;\ 0 \leq m \leq S\}$$

The generator of the chain is defined by equality (2), but here a transition from any state (n, m) to state $(n+1, m)$ is possible, since

the queue length is unlimited. Taking into account the system operation mechanism, the nonzero elements of the generator are determined similarly to relations (3)-(10). In particular, instead of relations (9) and (10), the following are taken, respectively:

$$q\big((n,m);(n_1,S)\big) = \nu_1 \cdot I(r < m \le s); \tag{43}$$

$$q\big((n,m);(n,S)\big) = \nu_2 \cdot I(0 \le m \le r). \tag{44}$$

Thus, the constructed 2DMC represents a level-independent quasi-birth-death process (LIQBD). If the states of this 2DMC are numbered in lexicographic order (i.e., $(0,0),(0,1),\dots,(0,S),$ $(1,0),(1,1),\dots,(1,S),\dots$) then the generator of the obtained LIQBD process is expressed in block-matrix form as:

$$G = \begin{pmatrix} B & A_0 & O & O & O & \cdots \\ A_2 & A_1 & A_0 & O & O & \cdots \\ O & A_2 & A_1 & A_0 & O & \cdots \\ O & O & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{45}$$

Here

$$B = \|b_{ij}\|,\ A_k = \left\|a_{ij}^{(k)}\right\|,\ k = 0,1,2;\ i,j = 0,1,\dots,S, \tag{46}$$

are square block matrices of dimension $(S+1)\times(S+1)$, whose nonzero elements are defined as follows:

$$b_{ij} = \begin{cases} \nu_2, & \text{if } 0 \le i \le r, j = S, \\ \nu_1, & \text{if } r < i \le s, j = S, \\ \kappa, & \text{if } 0 < i \le S, j = i-1, \\ -(\nu_2 + \lambda\phi_1), & \text{if } i = j = 0, \\ -(\nu_2 + \kappa + \lambda), & \text{if } 0 < i \le r, i = j, \\ -(\nu_1 + \kappa + \lambda), & \text{if } r < i \le s, i = j, \\ -(\kappa + \lambda), & \text{if } s < i \le S, i = j; \end{cases} \tag{47}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda\varphi_1, & \text{if } i = j = 0, \\ \lambda, & \text{if } i > 0, i = j; \end{cases} \tag{48}$$

$$a_{ij}^{(2)} = \begin{cases} \tau, & \text{if } i = j = 0, \\ \mu_1\sigma_1, & \text{if } i > 0, i = j, \\ \mu_2\sigma_2, & \text{if } i > 0, j = i-1. \end{cases} \tag{49}$$

$$a_{ij}^{(1)} = \begin{cases} \nu_2, & \text{if } 0 \le i \le r, j = S, \\ \nu_1, & \text{if } r < i \le s, j = S, \\ \kappa, & \text{if } 0 < i \le S, j = i-1, \\ -(\tau + \nu_2 + \lambda\phi_1), & \text{if } i = j = 0, \\ -(\nu_2 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } 0 < i \le r, i = j, \\ -(\nu_1 + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } r < i \le s, i = j, \\ -(\kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } s < i \le S, i = j; \end{cases} \quad (50)$$

For the system to operate in steady state, it is necessary that the underlying 2DMC be ergodic (stable).

The following theorem is proved:

Theorem. The system using the (s, S) replenishment policy is ergodic if and only if the following condition holds:

$$\lambda\big(1 - (1-\varphi_2)\pi(0)\big) < \tau\pi(0) + (\mu_1\sigma_1 + \mu_2\sigma_2)\big(1 - \pi(0)\big), \quad (51)$$

where $\pi(0)$ is defined as

$$\pi(0) = \big[(1+\theta_2)^r\big(1 + \theta_2((1+\theta_1)^{s-r} - 1 + (S-s)(1+\theta_1)^{s-r-1})\big)\big]^{-1}$$

$$\theta_i = \frac{\nu_i}{(\mu_2\sigma_2 + \kappa)}, i = 1,2.$$

Let the stationary distribution corresponding to generator G be denoted by

$$p = (p_0, p_1, p_2, \dots),$$

where

$$p_n = \big(p(n,0), p(n,1), \dots, p(n,S)\big), n = 0,1, \dots .$$

Based on the known LIQBD algorithm, it follows that if the ergodicity condition (51) holds, then the stationary distribution is determined as

$$p_n = p_0 R^n, \; n \ge 1, \quad (52)$$

where R is a nonnegative square matrix of dimension $(S+1)(S+1)$ and is the minimal solution of the following quadratic matrix equation:

$$R^2 A_2 + R A_1 + A_0 = 0 \quad (53)$$

The boundary probabilities p_0 are calculated from the following SGBE:

$$p_0(B + RA_2) = 0, \quad (54)$$

$$p_0(\boldsymbol{I} - R)^{-1}\boldsymbol{e} = 1, \quad (55)$$

where $\boldsymbol{I}$ and e are the identity matrix of order $(S+1)$ and the column vector of dimension $(S+1)$, respectively.

Thus, if the ergodicity condition (51) holds, the stationary probabilities of the constructed 2DMC are computed using equations (52)-(55).

The main performance measures of the system are defined similarly to those in the previous chapter, but it must be taken into account that in the considered models the queue length is infinite. Therefore, in the following formulas the upper limits of summation are taken as infinity.

• Average inventory level in the warehouse (S_{av}):

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n,m);$$

• Average order volume delivered by SS ($V_{av}(1)$):

$$V_{av}(1) = \sum_{m=r+1}^{S} (S-m) \sum_{n=0}^{\infty} p(n,m);$$

• Average order volume delivered by FS ($V_{av}(2)$):

$$V_{av}(2) = \sum_{m=0}^{r} (S-m) \sum_{n=0}^{\infty} p(n,m).$$

• Average number of *c*-customers in the system (L_{av}):

$$L_{av} = \sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m);$$

• Average destruction rate of inventory (*DRS*):

$$DRS = \kappa \left(1 - \sum_{n=0}^{\infty} p(n,0)\right);$$

• Average rate of orders to SS (RR_1):

$$RR_1 = \kappa p(0, s+1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^{\infty} p(n, s+1);$$

• Average rate of orders to FS (RR_2):

$$RR_2 = \kappa p(0, r+1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^{\infty} p(n, r+1);$$

• Loss probability of *c*-customers (*PL*):

$$PL = \varphi_2 \sum_{n=0}^{\infty} p(n,0) \ + \frac{\tau}{\tau + \lambda\varphi_1 + \nu_2} \sum_{n=1}^{\infty} p(n,0).$$

With the help of numerical experiments, the main system parameters are varied sequentially, and their influence on the main performance measures is systematically investigated. In all experiments,

$$\varphi_1 = 0.6,\ \sigma_1 = 0.4,\ \lambda = 20,\ \kappa = 10,\ \mu_1 = 35,\ \mu_2 = 25,$$
$$S = 22,\ \tau = 20,\ \nu_1 = 5,\ \nu_2 = 10$$

are taken as fixed.

Considering the limited length of the abstract, only the dependence of the performance measures on the *c*-customer arrival intensity is presented in the following table.

Table 5. Dependence of performance measures on parameter λ

λ	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	*DRS*	RR_1	RR_2	*PL*
20	2.39	0.76	14.49	2.22	9.94	1.32	0.44	0.09
22	2.43	0.84	14.41	3.13	9.93	1.38	0.48	0.13
24	2.47	0.91	14.32	4.91	9.92	1.43	0.52	0.18
26	2.50	0.98	14.25	8.60	9.90	1.48	0.56	0.23
28	2.54	1.06	14.16	30.26	9.89	1.54	0.61	0.30

In this chapter, mathematical models were also constructed for the (s, Q), $(S-1, S)$, and hybrid replenishment policies. In addition, the problem of minimizing total costs was considered, and by selecting the parameters s and r involved in the replenishment policies, the minimal total system costs were determined.

CONCLUSION

In the dissertation, the problems of mathematical modeling, analysis, and minimization of total costs in queuing-inventory systems supplied from two different sources have been solved. Based on the conducted research, the following main scientific and practical results have been obtained:

1. Queuing-inventory systems with replenishment from two different sources and having both finite and infinite buffer capacities

have been comprehensively studied. For the considered systems, four different replenishment policies – (s, Q), (s, S), hybrid and $(S-1, S)$ – have been proposed, and for each case the corresponding mathematical models have been constructed based on two-dimensional Markov chains. For each model, the state space has been determined and the generator matrices have been derived [1], [2], [3], [4], [7], [12].

2. For the study of queuing-inventory system models with finite buffer capacity, both an exact method based on solving the balance equations of two-dimensional Markov chains and an approximate method based on the space-merging algorithm for large-scale models have been developed. By applying the approximate method, analytical formulas determining the stationary probabilities and the main performance measures of the studied systems have been obtained. The proposed approximate approach makes it possible to calculate system performance measures without directly solving complex balance equations, and the high accuracy of the method has been confirmed by numerous numerical experiments [7], [10], [12].

3. For the study of queuing-inventory system models with infinite buffer capacity, Neuts' matrix-geometric method has been applied. Based on this approach, ergodicity conditions ensuring the long-term stable operation of the systems have been obtained, and an algorithmic method for computing the stationary distribution has been developed. The derived ergodicity conditions are expressed through relationships that can be easily verified using the system parameters. In addition, analytical formulas determining the main performance measures of the system have been obtained [1], [2], [3], [4].

4. For each model, explicit formulas have been derived to calculate the main performance and economic indicators that are important both from the customer's and the system manager's points of view. These formulas enable a comprehensive operational and economic evaluation of system performance. In particular, formulas have been obtained for calculating indicators such as the average inventory level, the intensity of lost demands, the average waiting time, and the average intensity of replenishment orders [3], [4], [5], [6], [7], [9], [12].

5. Based on the constructed models, the problems of minimizing total costs have been solved. For this purpose, the optimal values of the structural parameters of the considered replenishment policies have been determined, and the influence of system parameters on performance measures has been thoroughly analyzed through numerical experiments. The solved problems make it possible to select a control strategy that ensures a reasonable balance between the costs associated with inventory holding and replenishment orders and the costs arising from lost demands [3], [4], [7], [8], [11].

The main content of the dissertation has been published in the following scientific works

1. Mirzayev, R. On one model of a queuing-inventory system with two supply sources // Proceedings of the VII International Scientific-Practical Youth Conference "Creative Potential of Youth in Solving Aerospace Problems" (February Reports-2022), – Baku: National Aviation Academy, – 2022. – pp. 57-59. *(in Russian)*
2. Mirzayev, R. A model of a queuing system with a base stock policy and two supply sources // Informatics and Control Problems, – 2022. Vol. 42, no. 1. – pp. 26-32.
3. Melikov, A., Mirzayev, R., Nair, S. Double sources queuing-inventory system with hybrid replenishment policy // Mathematics, – 2022. Vol. 10, no. 14: 2423. doi: 10.3390/math10142423
4. Melikov, A., Mirzayev, R., Nair, S. Numerical study of double source queuing-inventory systems with destructive customers // Journal of Computer and Systems Sciences International, – 2022. Vol. 61, no. 4. – pp. 581-598.
5. Melikov, A., Mirzayev, R., Nair, S. Method for calculating the performance measures of a queuing system with a hybrid replenishment policy from two sources // Distributed Computer and Communication Networks (DCCN 2022): Proceedings of the XXV International Scientific Conference, – Moscow: – 2022. – pp. 143-149. *(in Russian)*
6. Mirzayev, R. Numerical analysis of the performance measures of a queuing-inventory system with two supply sources // Proceedings of the II Republican Scientific Conference "Fundamental Problems

of Mathematics and Application of Intelligent Technologies in Education", – Sumgayit: SSU, – 2022. – pp. 205-209. *(in Russian)*
7. Melikov, A., Mirzayev, R., Sztrik, J. Double-Sources Queuing-Inventory Systems with Finite Waiting Room and Destructible Stocks // Mathematics, – 2023. Vol. 11, no. 1, 226. doi:10.3390/math11010226
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10. Mirzayev, R., Aliyeva, S. Model of Double-Source Queuing-Inventory System with a Finite Buffer under Hybrid Replenishment Policy // International Conference on Management and Control in Solving Engineering Problems (MaCoSEP 2025), – Baku: – 2025. – 5 p. doi: 10.30546/MaCoSEP2025.1356
11. Mirzayev, R. Optimal Cost Analysis in a Double-Source Queuing-Inventory System under Hybrid Replenishment Policy // Journal of Baku Engineering University – Mathematics and Computer Science, – 2025. Vol. 9, no. 1. – pp. 19-25.
12. Mirzayev, R., Aliyeva, S. Hybrid Replenishment Policy in a Dual-Source Queuing-Inventory System with Finite Buffer // Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie vychislitelnaja tehnika i informatika – Tomsk State University Journal of Control and Computer Science, – 2026. №74. –p.71-80.

Personal contribution of the applicant in co-authored publications:

[3], [4], [7] – participation in software development, model validation, analysis, methodology development, and in the stages of editing and improving the manuscript.
[5] – development of software using exact and approximate formulas for calculating stationary probabilities and performance measures, evaluation of the accuracy of the approximate formulas through numerical experiments, and processing of the obtained results.
[10], [12] – formulation of the problem, mathematical derivation of the formulas, performance of computations, obtaining and analysis of the results, as well as preparation of the main text of the manuscript.

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