Investigation in Stability of Markowitz's Multicriterial Portfolio Optimization Problem with Wald's Maximin Criteria in Euclidean Metric

Vladimir Emelichev[1], Vladimir Korotkov[2]
Belarusian State University, Minsk, Belarus
[1]*emelichev@bsu.by*, [2]*wladko@tut.by*

***Abstract*— The vector Boolean variant of well-known Markowitz's investment problem is considered with Wald's maximin efficient criteria. Lower and upper attainable bounds for the stability radius of a Pareto-optimal portfolio of the problem are obtained in the case with Euclidian metric in the problem parameters space.**

Keywords*— *multicriteria investment problem; Pareto-optimal investment portfolio; portfolio efficiency; Wald's maximin criteria; stability radius; stability

Based on Markowitz's portfolio theory [1] we consider s-criteria discrete (Boolean) variant of the investment managing problem with Wald's maximin criteria [2]

$$f_k(x,E_k)=\min_{i\in N_m}\sum_{j\in N_n} e_{ijk}x_j \to \max_{x\in X}, \qquad k\in N_s. \qquad (1)$$

Here $N_n=\{1,2,\dots,n\}$ is the set of alternative investment projects (assets); N_m is the set of possible market states (market situations, scenarios); N_s is the set of efficiency measures; $x=(x_1,x_2,\dots,x_n)$ is an investment portfolio, where $x_j=1$ is, if investment project $j\in N_n$ is implemented, and $x_j=0$ otherwise; e_{ijk} is an expected efficiency for measure $k\in N_s$ of investment project $j\in N_n$ in the case, when the market was in state $i\in N_m$; $X\subseteq\{0,1\}^n$ is the set of investment portfolios.

Thus initial date is matrix $E=[e_{ijk}]\in \mathbf{R}^{m\times n\times s}$, composed of efficiency assessment of portfolios. Note, that there exist various approaches to assessing investment projects (NPV, NFV, PI etc.) which take into account the impact of uncertainty and risk in different ways (see for example [3–6]). It is known that the complexity of calculation of such quantities, application of statistical and expert project efficiency estimates, is accompanied by a large number of errors resulting in high degree of uncertainty of initial information [7]. In this relation, the question on the limiting variation (perturbation) level for parameters of the initial problem preserving the optimality of the chosen portfolio naturally arises. This quantitative approach results in the key idea of the stability radius. The other type of uncertainty, arising in the choice of the most efficiency portfolio, is associated with instability and unpredictability of the market state. Using Wald's maximin criteria we take it into account. Obviously, that the investor, following Wald's criteria, takes extreme caution and chose the maximum total portfolio efficiency, assuming that the market was in the worst state, namely the efficiency is minimal.

A problem (1) means the problem of searching the Pareto set, i.e. the set of Pareto-optimal investment portfolios of this problem

$$P^s(E)=\{x\in X:\ \nexists x'\in X$$
$$(f(x',E)\geq f(x,E)\ \&\ \ f(x',E)\neq f(x,E))\},$$

where

$$f(x,E)=(f_1(x,E_1),f_2(x,E_2),\dots,f_s(x,E_s)).$$

In the real space $\mathbf{R}^d$ of arbitrary dimension $d\in\mathbf{N}$ we introduce Euclidean metric l_2, that is the norm of $a=(a_1,a_2,\dots,a_d)\in\mathbf{R}^d$ is meant by the number $\|a\|_2=\sqrt{\sum_{i\in N_d}|a_i|}$, and by the norm of a matrix is meant the norm of the vector composed of all matrix elements.

The stability radius of Pareto-optimal portfolio $x^0\in P^s(E)$ of s-criteria problem (1), as usual [8, 9], is defined as follows:

$$\rho(x^0,m,s)=\begin{cases}\sup\Xi, & \text{if } \ \Xi\neq\varnothing,\\ 0, & \text{if } \ \Xi=\varnothing,\end{cases}$$

where

$$\Xi=\{\varepsilon>0:\forall E'\in\Omega(\varepsilon)\ \ (x^0\in P^s(E+E'))\},$$

$$\Omega(\varepsilon)=\{E'\in\mathbf{R}^{m\times n\times s}:\ \|E'\|_2<\varepsilon\},\quad E'=[e'_{ijk}].$$

Thus, the stability radius defines an extreme level of problem initial data perturbations (elements of matrix E) preserving Pareto-optimality of the portfolio.

Theorem. *For any* m, $s \in \mathbf{N}$ *the stability radius* $\rho(x^0, m, s)$ *of any Pareto-optimal portfolio* $x^0 \in P^s(E)$ *has the following lower and upper bounds*

$$\varphi(x^0, m, s) \le \rho(x^0, m, s) \le \psi(x^0, m, s)\sqrt{m},$$

where

$$\varphi(x^0, m, s) = \min_{x \in X \setminus \{x^0\}} \frac{\left\| [f(x^0, E) - f(x, E)]^+ \right\|_2}{\left\| x^0 \right\|_2 + \left\| x \right\|_2},$$

$$\psi(x^0, m, s) = \min_{x \in X \setminus \{x^0\}} \frac{\left\| [f(x^0, E) - f(x, E)]^+ \right\|_2}{\left\| x^0 - x \right\|_2}.$$

Here $[z]^+ = (z_1^+, z_2^+, \dots, z_s^+)$, $z_k^+ = \max\{z_k, 0\}$, $k \in N_s$, $z = (z_1, z_2, \dots, z_s) \in \mathbf{R}^s$ is.

Corollary 1. *If for any portfolio* $x \in X \setminus \{x^0\}$ *the equation* $\left\| x^0 \right\|_2 + \left\| x \right\|_2 = \left\| x^0 - x \right\|_2$ *holds, then for any index* $s \in \mathbf{N}$ *the following formula is true:*

$$\rho(x^0, 1, s) = \varphi(x^0, 1, s) = \psi(x^0, 1, s).$$

Thus, lower and upper bounds are attainable for $m = 1$.

Corollary 2. *There exists a class of problems (1), such that for any* m, $s \in \mathbf{N}$ *the following formula is valid:*

$$\rho(x^0, m, s) = \varphi(x^0, m, s).$$

This corollary indicates the attainability of lower bound of the stability radius to establish Theorem for any parameters m and s.

Note that earlier in [10] the similar bounds were obtained for the stability radius of an optimal portfolio of the multicriteria investment Boolean problem with Savage's ordered risk criteria, and in [9] were announced and in [11] were published attainable bounds of the stability radius of the multicriteria investment problem with Savage's risk criteria and the Pareto optimality principle in the case of Chebyshev l_∞ metric in the three-dimensions problem parameters space.

REFERENCES

[1] H.M. Markowitz. Portfolio selection: efficient diversification of investments. Oxford: Blackwell, 1991.

[2] A. Wald. Statistical decision functions. New York: Wiley, 1950.

[3] E.M. Bronshtein, M.M. Kachkaeva, E.V. Tulupova, "Control of investment portfolio based on complex quantile risk measures," Journal of Computer and System Sciences International, vol. 50, N. 1, pp. 174–180, 2011.

[4] P.L. Vilensky, V.N. Livshiz, S.A. Smolyk. Evaluation of investment project effectiveness. Moscow: Theory and Practice, 2008.

[5] V.V. Carev. Assessing the economic efficiency of investments. Saint Petersburg: Piter, 2004.

[6] Portfolio decision analysis: improved methods for resource allocation (International series in operations research and management science). A. Salo, J. Keisler, A. Morton, Eds.. New York: Springer, 2011.

[7] A. N. Shiryaev. Essentials of stochastic finance: facts, models, theory. Singapore: World Scientific, 1999.

[8] V.A. Emelichev, V.V. Korotkov, K.G. Kuz'min, "Multicriteria investment problem in conditions of uncertainty and risk," Journal of Computer and System Sciences International, vol. 50, N. 6, pp. 1011–1018, 2011.

[9] V. Emelichev, V. Korotkov, "Investment optimization problem with savage's risk criteria under uncertainties," Problems of Cybernetics and Informatics (PCI'2010): papers of the third international conference, vol. III. Baku. Azerbaijan. September 6–8, 2010. Baku: AzNAS, pp. 20–23, 2010.

[10] V.A. Emelichev, V.V Korotkov, "On stability of optimal portfolio of Markowitz's investment problem with ordered Savage's risk criteria," Transaction of AzNAS, Series of Phys.-Tech. and Math. Sciences, vol. 31, N. 6, pp. 27–34, 2011.

[11] V. Emelichev, V. Korotkov, K. Kuzmin, "On stability of a Pareto-optimal solution of a portfolio optimization problem with Savage's minimax risk criteria," Bulletin of the Academy of Sciences of Moldova. Mathematics, N. 3 (64), pp. 35–44, 2010.