Optimal Control for Schrodinger's Equation with Pure Imaginary Coefficient in the Nonlinear Part of the Equation

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Abstract— **Work is devoted study of a problem of optimal control for a Schrodinger equation with purely imaginary factor in a nonlinear part of this equation where control is quadratically the summable function, and the criterion of quality is a functional of Lions. With that end in view at first the correctness of statement of the reduced problem is investigated and the correctness of statement of a problem of optimum control is studied. Differentiability of a functional of Lions is investigated and the necessary condition of an optimality in the form of a variation inequality is established.**

Keywords— Schrodinger equation; optimum control; criterion of Lions

I. INTRODUCTION

Optimal control problems for Schrodinger's nonlinear equation often arise in quantum mechanics, nuclear physics, nonlinear optics, super conductivity theory and in other fields of up to-date physics and engineering, in which the coefficient of this equation plays as a control [1,2]. Optimal control problems for Schrodinger's nonlinear equation were previously investigated, for example, in the papers [3-7], and others where control functions are bounded and measurable functions.

In the paper we consider an optimal control problem for Schrodinger's equation with pure imaginary coefficient in the nonlinear part of the equation with Lion's quality test, where a square-summable function is a control. It should be noted that optimal control problems for linear and nonlinear Schrodinger's equations, with square-summable control were studied earlier for example, in the papers [8-11] and others. Pay attention to the fact that by the statement and obtained results the given paper differs from the earlier studied ones.

II. PROBLEM STATEMENT

Let $l>0$, $T>0$ be given numbers, $x\in(0,l)$, $t\in[0,T]$, $\Omega=(0,l)\times(0,t)$. Let $L_p(\Omega)$ be a Lebesgue space of measurable functions summable over the power $p\ge 1$, $C^k([0,T],B)$ be a Banach space consisting of all definite, $k\ge 0$ times continuously differentiable on $[0,T]$ functions with values in the Banach space B, $W_p^k(0,l)$, $W_k^{k,m}(\Omega)$ be Sobolev spaces [12,13] of functions with generalized derivatives of order $k\ge 0$ with respect to x and order $m\ge 0$ from t, respectively, that are summable over the power $p\ge 1$, $\overset{0}{W}{}_2^1(0,l)$ be a subspace of the space $W_2^1(0,l)$, whose elements vanish at the ends of the segment $[0,l]$. $\overset{0}{W}{}_2^2(0,l)=W_2^2(0,l)\cap\overset{0}{W}{}_2^1(0,l)$.

Let's consider a problem on minimization of the functional

$$J_\alpha(v)=\int_\Omega|\psi_1(x,t)-\psi_2(x,t)|^2dxdt+\alpha\|v-\omega\|_H^2 \quad (1)$$

on the set $V\equiv\{v=v(x): v\in W_2^1(0,l), \|v\|_{W_2^1(0,l)}\le b\}$ under conditions:

$$i\frac{\partial\psi_k}{\partial t}+a_0\frac{\partial^2\psi_k}{\partial t}-a(x)\psi_k-v(x)\psi_k+ \\ +ia_1|\psi_k|^2\psi_k=f_k(x,t),\ (x,t)\in\Omega \quad (2)$$

$$\psi_k(x,0)=\psi_k(0),\ k=1,2,\ x\in(0,l) \quad (3)$$

$$\psi_1(0,t)=\psi_1(l,t)=0,\ t\in(0,T), \quad (4)$$

$$\frac{\partial\psi_2(0,t)}{\partial x}=\frac{\partial\psi_2(l,t)}{\partial x}=0,\ t\in(0,T), \quad (5)$$

where $i^2=-1$, $a_0>0$, $a_1>0$, $b>0$, $\alpha\ge 0$ are the given numbers, $H\equiv W_2^1(0,l)$, $\omega\in H$ is a given element, $a=a(x)$ is a bounded, measurable function satisfying the condition

$$0<\mu_0\le a(x)\le\mu_1,\ \left|\frac{da(x)}{dx}\right|\le\mu_2,$$

$$\forall x \in (0, l),\ \mu_0,\ \mu_1,\ \mu_2 = const > 0, \tag{6}$$

and the functions $\varphi_k(x),\ f_k(x,t),\ k=1,2$ satisfy the conditions:

$$\varphi_1 \in \overset{0}{W}{}_2^2(0,l),\quad \varphi_2 \in W_2^2(0,l),\quad \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0, \tag{7}$$

$$f_1 \in \overset{\circ}{W}{}_2^{1.1}(\Omega),\ f_2 \in W_2^{1.1}(\Omega. \tag{8}$$

The problem on definition of functions $\psi_k = \psi_k(x,t),\ k=1,2$ from conditions (2)-(5) for the given $v \in V$ is said to be a reduced problem. Under the solution of this problem we'll understand the functions $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v),\ k=1,2$, belonging to

$B_1 \equiv C^0\left([0,T], \overset{\circ}{W}{}_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$ and

$B_2 \equiv C^0([0,T], W_2^2(0,l)) \cap C^1([0,T], L_2(0,l))$ respectively and satisfying the conditions (2)-(5) for almost all $x \in (0,l)$ and $\forall t \in [0,T]$. As is seen, the reduced problem consists of two boundary value problems, i.e. the first and second boundary value problems for Schrodinger's equation with pure imaginary coefficients in the linear part of the equation. It should be noted that boundary value problems for Schrodinger's linear and nonlinear equation of kind (2) earlier were studied in the papers [3-8,11,14,-16]. However, these results are not sufficient for our goal, since in the indicated papers a wide class of functions is a class of bounded and measurable functions possessing generalized derivatives from $L_\infty(0,l)$. Therefore, these arises necessity at first to study the well-posedness of the statement of the reduced problem (2)-(5), with a coefficient from the set $V \subset W_2^1(0,l)$. Allowing for this remark by means of Galerkin's method and the proof methods of the papers [3-8,11,12] we proved the following statement:

Theorem 1. Let the functions $a(x),\ \varphi_k(x),\ f_k(x,t),\ k=1,2$ satisfy the conditions (6)-(8). Then the reduced problem (2)-(5) for each $v \in V$ has a unique solution $\psi_1 \subset B_1$ and $\psi_2 \subset B_2$ and the estimations [17]:

$$\|\psi_1(\cdot,t)\|_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial \psi_1(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le c_1\left(\|\varphi_1\|_{\overset{\circ}{W}{}_2^2(0,l)}\right) + c_1\left(\|f_1\|_{\overset{\circ}{W}{}_2^{1.1}(\Omega)} + \|\varphi_1\|^3_{\overset{\circ}{W}{}_2^1(0,l)} + \|f_1\|^3_{\overset{\circ}{W}{}_2^{1,0}(\Omega)}\right) \tag{9}$$

$$\|\psi_2(\cdot,t)\|_{W_2^2(0,l)} + \left\|\frac{\partial \psi_2(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \le c_2\left(\|\varphi_2\|_{W_2^2(0,l)}\right) + c_2\left(\|f_2\|_{W_2^{1.1}(\Omega)} + \|\varphi_2\|^3_{W_2^1(0,l)} + \|f_2\|^3_{W_2^{1.0}(\Omega)}\right) \tag{10}$$

are valid for $\forall t \in [0,T]$, where $c_1 > 0,\ c_2 > 0$ are some constants independent of t.

Theorem 2. Let all the conditions of theorem 1 be fulfilled and $\omega \in W_2^1(0,l)$ be a given element. Then there exists an everywhere dense subset G of the space $W_2^1(0,l)$ such that for any $\omega \in G$ at $\alpha > 0$ the optimal control problem (1)-(5) has a unique solution.

Theorem 3. Let the conditions of theorem 2 be fulfilled and $\alpha \ge 0$ be a given number. Then the optimal control problem (1)-(5) has at least one solution.

Let's view the following adjoint problem an definition of functions $\eta_k = \eta_k(x,t),\ k=1,2$ from the conditions.

$$i\frac{\partial \eta_k}{\partial t} + a_0\frac{\partial^2 \eta_k}{\partial x^2} - a(x)\eta_k - v(x)\eta_k - ia_1\left(2|\psi_k|^2\eta_k - \psi_k^2\bar{\eta}_k\right) = 2(-1)^k(\psi_1(x,t) - \psi_2(x,t)),\ (x,t) \in \Omega, \tag{11}$$

$$\eta_k(x,T) = 0,\quad x \in (0,l),\ k = 1,2, \tag{12}$$

$$\eta_1(0,T) = \eta_1(l,t) = 0,\quad t \in (0,T), \tag{13}$$

$$\frac{\partial \eta_2(0,t)}{\partial x} = \frac{\partial \eta_2(l,t)}{\partial x} = 0,\quad t \in (0,T), \tag{14}$$

where $\psi_k = \psi_k(x,t),\ k=1,2$ is a solution of the reduced problem (2)-(5) for $v \in V$.

Under the solution of the adjoint problem we'll understand the functions $\eta_k = \eta_k(x,t),\ k=1,2$ from the space $C^0([0,T], L_2(0,l))$, satisfying the integral identities:

$$\int_\Omega \left\{\eta_k\left[-i\frac{\partial \bar{\Phi}_k}{\partial t} + a_0\frac{\partial^2 \bar{\Phi}_k}{\partial x^2} - a(x)\bar{\Phi}_k - v(x)\bar{\Phi}_k\right] - \eta_k 2a_1 i|\psi_k|^2\bar{\Phi}_k + ia_1\bar{\eta}_k\psi_k^2\bar{\Phi}_k\right\}dxdt = 2(-1)^k\int_\Omega(\psi_1(x,t) - \psi_2(x,t))\bar{\Phi}_k(x,t)dxdt, \tag{15}$$

where $k=1,\ 2$.

For any functions $\Phi_1 \in \overset{\circ}{W}{}_2^{2,1}(\Omega)$, $\Phi_2 \in W_2^{2,1}(\Omega)$ satisfying the conditions:

$$\frac{\partial \Phi_2(0,t)}{\partial x} = \frac{\partial \Phi_2(l,t)}{\partial x} = 0, \quad t \in (0,T),$$
$$\Phi_k(x,0) = 0, \qquad k = 1,2 \quad .$$

Theorem 4. Let the conditions of theorem be fulfilled and $\omega \in W_2^1(0,\ l)$ be a given element. Then for any function $w = w(x)$ from $W_2^1(0,\ l)$ it is valid the following expression for the first variation of the functional $J_\alpha(v)$:

$$\delta J_\alpha(v,w) = \int_0^l \Big[2\alpha\big(v(x) - \omega(x)\big)w(x) -$$
$$-\int_0^T \operatorname{Re}\big(\psi_1(x,t)\overline{\eta}_1(x,t) + \psi_2(x,t)\overline{\eta}_2(x,t)\big)dt \cdot w(t) +$$
$$+2\alpha\left(\frac{dv(x)}{dx} - \frac{d\omega(x)}{dx}\right)\frac{dw(x)}{dx}\Big]dx, \qquad (16)$$

where $\psi_k = \psi_k(x,\ t) \equiv \psi_k(x,\ t;\ v)$, $\eta_k = \eta_k(x,\ t) \equiv \eta_k(x,\ t;\ v)$, $k=1,\ 2$, are the solutions of the reduced and adjoint problems for $v \in V$.

Theorem 5. Now let the conditions of theorem 4 be fulfilled and $v^* = v^*(x)$ from V be an optimal control in the problem (1)-(5). Then for $\forall v \in V$ the following inequality:

$$\int_0^l \left\{\left[\int_0^T \operatorname{Re}\big(\psi_1^*(x,t)\overline{\eta}_1^*(x.t) + \psi_2^*(x,t)\big)dt -\right.\right.$$
$$\left.-2\alpha\big(v^*(x) - \omega(x)\big)\right]\big(v(x) - \omega^*(x)\big) - 2\alpha\, *$$
$$\left. * \left(\frac{dv^*(x)}{dx} - \frac{d\omega(x)}{dx}\right)\left(\frac{dv(x)}{dx} - \frac{dv^*(x)}{dx}\right)\right\}dx \le 0, \quad (17)$$

is fulfilled, where $\psi_k^*(x,\ t) = \psi_k(x,\ t;\ v^*)$ and $\eta_k^*(x,t) = \eta_k(x,t;v^*)$, $k = 1,\ 2$ are the solutions of the reduced and ajoint problem for $v^* \in V$.

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