About Extremum Problems with Constraints in a Metric Space

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Abstract— **In the work, we define $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz functions at a point in a metric space. We also study a number of their properties and consider extremum problems with constraints. Using distance functions, we obtain a number of theorems concerning the exact penalty. We obtain necessary extremum conditions under some constraints in a Banach space.**

Keywords— extremal problem; Lipschitz function; metric space; Banach space

I. Introduction

The prominent role that nonsmooth analysis plays in connection with optimization theory is widely recognized, especially since the latter has natural mechanisms that generate nonsmoothness: dualty theory, sensitivity and stability analysis, mechanics, economics, etc.

In work [1] we have defined class $\varphi-(\alpha,\beta,\nu,\delta)$ of Lipschitz functions in point and usinq that function we have obtained high order necessary conditions of extremum for extremal problem with constraints. We also study a number of their properties and using distance functions, we obtain a number of theorems concerning the exact penalty.

II. Properties of $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz Functions

Let (X,d) be a metric space, G, $C\subset X$, $f\colon X\to R$, $\alpha>0$, $\nu>0$, $\beta\geq\alpha\nu$, $\delta>0$ and $\omega: R_+\to R_+$, where $\omega(0)=0$, $R_+=[0,+\infty)$. Let us assume $B(x,\delta)=\{y\in X: d(x,y)\leq\delta\}$.

The function f is called a $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz function with the constant K at the point $\bar{x}$ if f satisfies the following condition

$$|f(y)-f(x)|\leq Kd(x,y)^{\nu}\left(d(x,\bar{x})^{\beta-\alpha\nu}+d(x,y)^{\frac{\beta-\alpha\nu}{\alpha}}\right)+\omega(d(x,\bar{x}))$$

for $x,y\in B(\bar{x},\delta)$. If $\omega(t)\equiv 0$, then the function f is called a $(\alpha,\beta,\nu,\delta)$- Lipschitz function with the constant K at the point $\bar{x}$.

The function f is called a $(\alpha,\beta,\nu,\omega)$-Lipschitz function with the constant K at the point $\bar{x}\in G$ with respect to the set G if f satisfies the following condition

$$|f(y)-f(x)|\leq Kd(x,y)^{\nu}\left(d(x,\bar{x})^{\beta-\alpha\nu}+d(x,y)^{\frac{\beta-\alpha\nu}{\alpha}}\right)+\omega(d(x,\bar{x}))$$

for $x,y\in G$. If $\omega(t)\equiv 0$, then the function f is called a (α,β,ν)-Lipschitz function with the constant K at the point $\bar{x}\in G$ with respect to the set G.

The set of all functions $f: B(\bar{x},\delta)\to R$ satisfying the $(\alpha,\beta,\nu,\delta)$-Lipschitz condition at the point $\bar{x}$ is denoted by $L(\alpha,\beta,\nu,\delta)$.

Lemma 1.1. $L(\alpha,\beta,\nu,\delta)$ is a normalized space with respect to the following norm

$$\|f\|_L=\sup_{\substack{x,y\in B(\bar{x},\delta)\\ y\neq x}}\frac{|f(y)-f(x)|}{d(x,y)^{\nu}\left(d(x,\bar{x})^{\beta-\alpha\nu}+d(x,y)^{\frac{\beta-\alpha\nu}{\alpha}}\right)}+|f(\bar{x})|.$$

If $f\in L(\alpha,\beta,\nu,\delta)$, then from the definition of the $(\alpha,\beta,\nu,\delta)$- Lipschitz functions it follows that f is continuous at the set $B(\bar{x},\delta)$.

The function f is called a (θ,δ,ω)- bilipschitz function with the constant K at the point $\bar{x}$, if f satisfies the following condition

$$|f(y)-f(x)|\leq Kd(x,y)^{\theta}(d(x,\bar{x})+d(y,\bar{x}))^{2-\theta}+\omega(d(x,\bar{x}))$$

for $x,y\in B(\bar{x},\delta)$, $\delta>0$, and $0<\theta\leq 2$.

Lemma 1.2. If f satisfies the (θ,δ,ω)-bilipschitz condition at the point $\bar{x}$, then f satisfies the $(1,2,\theta,\delta,\omega)$-Lipschitz condition at the point $\bar{x}$. Conversely, if f satisfies the $(1,2,\theta,\delta,\omega)$-Lipschitz condition at the point $\bar{x}$, then f satisfies the (θ,δ,ω)-bilipschitz condition at the point $\bar{x}$.

Lemma 1.3. If the function f_τ, $\tau\in\Omega$, satisfies the $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz condition with the constant L_τ at the point $\bar{x}$ and $L=\sup\{L_\tau:\tau\in\Omega\}<+\infty$, then $f(x)=\sup_{\tau\in\Omega} f_\tau(x)$ satisfies the $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz condition with the constant L at the point $\bar{x}$ as well.

The function f is called a strongly $(\alpha,\beta,\nu,\delta)$-Lipschitz function with the constant K at the point $\bar{x}$ if there exists a function $\omega:R_+\to R_+$, where $\lim_{t\downarrow 0}\frac{\omega(t)}{t}=0$, such that f satisfies the following condition:

$$|f(y)-f(x)|\le Kd(x,y)^\nu\left(d(x,\bar{x})^{\beta-\alpha\nu}+d(x,y)^{\frac{\beta-\alpha\nu}{\alpha}}\right)+\omega(d(x,\bar{x})^\beta)$$

for $x,y\in B(\bar{x},\delta)$.

The function f is called a conditionally $(\alpha,\beta,\nu,\delta)$-Lipschitz function with the constant K at the point $\bar{x}$ if there exists a function $\omega:R_+\to R_+$, where $\lim_{t\downarrow 0}\frac{\omega(t)}{t}=0$, such that f satisfies the following condition

$$|f(y)-f(x)|\le Kd(x,y)^\nu\left(d(x,\bar{x})^{\beta-\alpha\nu}+d(x,y)^{\frac{\beta-\alpha\nu}{\alpha}}\right)+\omega(d(x,\bar{x})^\beta)$$

for $x,y\in B(\bar{x},\delta)$, $d(y,\bar{x})\le d(x,\bar{x})$.

III. Minimization Problems in a Metric Space

Let (X,d) be a metric space, G, $C\subset X$, $f:X\to R$, $\alpha>0$, $\nu>0$, $\beta\ge\alpha\nu$, $\delta>0$, and $\omega:R_+\to R_+$, where $\omega(0)=0$. Let us assume $d(x)=d_C(x)=\inf\{d(x,y):y\in C\}$.

Theorem 2.1. Let x0 be a minimum point of the function f at the set C, and f satisfies the $(\alpha,\beta,\nu,\omega)$-Lipschitz condition at the point x_0 with respect to the set G with the constant K and $C\subset G$. Then for any $\lambda\ge K$ the function

$$g_\lambda(x)=f(x)+\lambda\left(d^{\frac{\beta}{\alpha}}(x)+d(x,x_0)^{\beta-\alpha\nu}d^\nu(x)\right)+\omega(d(x,x_0))$$

reaches at G its minimal value at the point $x_0\in G$, and if $\lambda>K$, then any point minimizing $g\lambda(x)$ at the set G belongs to C, where $d^\nu(x)=d(x)^\nu$.

Corollary 2.1. Let x_0 be a minimum point of the function f at the set C, and f satisfies the $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz condition at the point x_0 with the constant K and $C\subset B(x_0,\delta)$. Then for any $\lambda\ge K$ the function

$$g_\lambda(x)=f(x)+\lambda\left(d^{\frac{\beta}{\alpha}}(x)+d(x,x_0)^{\beta-\alpha\nu}d^\nu(x)\right)+\omega(d(x,x_0))$$

reaches at $B(x_0,\delta)$ its minimal value at the point $x_0\in B(x_0,\delta)$, and if $\lambda>K$, then any point minimizing $g_\lambda(x)$ at $B(x_0,\delta)$ belongs to C.

Theorem 2.2. Let x_0 be a minimum point of the function f at the set $\{x\in C:g_i(x)\le 0,\ i=\overline{1,n}\}$, where $g_i:X\to R$, the functions f and g_i, $i=\overline{1,n}$, satisfy the $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz condition at the point $x_0\in C$ with the constant K, $C\subset B(x_0,\delta)$. Then for any $\lambda\ge K$ the function

$$g_\lambda(x)=$$

$$\max\left\{r_0(f(x)-f(x_0))+\sum_{i=1}^n r_i g_i(x):r_i\ge 0,\ i=\overline{0,n},\ \sum_{i=0}^n r_i=1\right\}^+$$

$$+\lambda\left(d^{\frac{\beta}{\alpha}}(x)+d(x,x_0)^{\beta-\alpha\nu}d^\nu(x)\right)+\omega(d(x,x_0))$$

reaches at $B(x_0,\delta)$ its minimal value at the point $x_0\in B(x_0,\delta)$, and if $\lambda>K$, then any point minimizing $g\lambda(x)$ at $B(x_0,\delta)$ belongs to C.

Theorem 2.3. Let x_0 be a minimum point of the function f at the set $\{x\in C:g_i(x)\le 0,\ i=\overline{1,n}\}$, where $g_i:X\to R$, and functions f and g_i $i=\overline{1,n}$, satisfy the $(\alpha,\beta,\nu,\delta,\omega)$-Lipschitz condition at the point $x_0\in C$ with the constant K, $C\subset B(x_0,\delta)$. Then for any $\lambda\ge K$ the function

$$g_\lambda(x)=\max\{f(x)-f(x_0),g_1(x),\dots,g_n(x)\}+$$
$$\lambda\left(d^{\frac{\beta}{\alpha}}(y)+d(y,x_0)^{\beta-\alpha\nu}d^\nu(y)\right)+\omega(d(y,x_0))$$

reaches at $B(x_0,\delta)$ its minimal value at the point $x_0\in B(x_0,\delta)$, and if $\lambda>K$, then any point minimizing $g_\lambda(x)$ at $B(x_0,\delta)$ belongs to C.

Let X and Y be metric spaces, $U\subset X$ be an open set. If for any $B_X(x,r)\subset U$ there holds true the inclusion $F(B_X(x,r))\supset B_Y(F(x),ar)$, then it is said that the operator F covers the open set U with the constant $a > 0$. Let us assume that $B_\varepsilon(\Omega) = \{x \in X : d(x,\Omega) \le \varepsilon\}$ is an closed ε-neighborhood of the set Ω.

Theorem 2.4. Let X be a full metric space, Y be a Banach space, U be an open set in X, the continuous mapping $F : U \to Y$ cover the open set U with the constant $a > 0$, $\Omega \subset U$ be a bounded set, there exist a number $\varepsilon > 0$ such that $B_\varepsilon(\Omega) \subset U$ and $\{x \in U : F(x) = 0\} = \{x \in \Omega : F(x) = 0\}$, the functions $f_i : U \to R$, $i = 0,1,\ldots,n$, satisfy the $(\alpha,\beta,\nu,\omega)$-Lipschitz condition at the point $x_0 \in C$, where $C = \{x \in \Omega : F(x) = 0\}$, with respect to the set Ω with the constant K, and x0 be a minimum point of the function f_0 at the set $\{x \in \Omega : f_i(x) \le 0,\ i = \overline{1,n},\ F(x) = 0\}$. Then there exists $m_0 > 0$ such that for any $\mu \ge m_0$ the function

$$g_\mu(x) = \max\{f_0(x) - f_0(x_0), f_1(x), \ldots, f_n(x)\} + \mu\left(\|F(x)\|^{\frac{\beta}{\alpha}} + d(x,x_0)^{\beta-\alpha\nu}\|F(x)\|^{\nu}\right) + \omega(d(x,x_0))$$

reaches at Ω its minimal value at the point $x_0 \in \Omega$ and any point minimizing $g_\mu(x)$ at Ω belongs to C.

Theorem 2.5. Let x_0 be a minimum point of the function f at the set $\{x \in C : g_i(x) \le 0,\ i = \overline{1,n}\}$, where $g_i : X \to R$, the functions f and g_i, $i = \overline{1,n}$, satisfy the $(\alpha,\beta,\nu,\delta)$-Lipschitz condition at some point $\overline{x} \in C$ with the constant K, $C \subset B(\overline{x}, \delta)$. Then for any $\lambda \ge K$ the function

$$g_\lambda(x) = \max\left\{r_0(f(x) - f(x_0)) + \sum_{i=1}^{n} r_i g_i(x) : r_i \ge 0,\ i = \overline{0,n},\ \sum_{i=0}^{n} r_i = 1\right\} + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d(x,\overline{x})^{\beta-\alpha\nu} d^{\nu}(x)\right)$$

reaches at $B(\overline{x}, \delta)$ its minimal value at the point $x_0 \in B(\overline{x}, \delta)$, and if $\lambda > K$, then any point x_0 minimizing g λ(x) at $B(\overline{x}, \delta)$ belongs to C.

Theorem 2.6. Let x_0 be a minimum point of the function f_0 at the set $\{x \in C : f_i(x) \le 0,\ i = \overline{1,n}\}$, where the functions f_0 and $f_i : X \to R$, $i = \overline{1,n}$, satisfy the $(\alpha,\beta,\nu,\delta)$-Lipschitz condition at some point $\overline{x} \in C$ with the constant K, $C \subset B(\overline{x}, \delta)$. Then for any $\lambda \ge K$ the function

$$g_\lambda(x) = \max\{f_0(x) - f_0(x_0), f_1(x), \ldots, f_n(x)\} + \lambda\left(d^{\frac{\beta}{\alpha}}(x) + d(x,\overline{x})^{\beta-\alpha\nu} d^{\nu}(x)\right)$$

reaches at $B(\overline{x}, \delta)$ its minimal value at the point $x_0 \in B(\overline{x}, \delta)$ and if $\lambda > K$, then any point minimizing $g_\lambda(x)$ at $B(\overline{x}, \delta)$ belongs to C.

Let X be a metric space, $f : X \to \overline{R} = R \cup \{\pm\infty\}$, $\mathrm{dom}f = \{x \in X : |f(x)| < +\infty\}$, and $x_0 \in \mathrm{dom}f$. Let us assume

$$f^+(x_0) = \overline{\lim_{x \to x_0}} \frac{f(x) - f(x_0)}{d(x,x_0)}$$

$$f^-(x_0) = \underline{\lim_{x \to x_0}} \frac{f(x) - f(x_0)}{d(x,x_0)}.$$

Theorem 2.7. Let X be a metric space, x0 be a minimum point of the function f at the set C, f satisfy the Lipschitz condition in the δ-neighborhood of the point x_0 with the constant K, and $C \subset B(x_0,\delta)$. Then $f^-(x_0) + \lambda d^+(x_0) \ge 0$ and $f^+(x_0) + \lambda d^-(x_0) \ge 0$ for any $\lambda \ge K$.

Let X be a Banach space. Let us assume $d_2(x) = d^2(x)$. If $f : X \to R$, then assume

$$f^+(x_0;x) = \overline{\lim_{t \downarrow 0}} \frac{f(x_0 + tx) - f(x_0)}{t}$$

$$f^-(x_0;x) = \underline{\lim_{t \downarrow 0}} \frac{f(x_0 + tx) - f(x_0)}{t}$$

$$f^{(2)-}(x_0;x) = \underline{\lim_{t \downarrow 0}} \frac{f(x_0 + tx) - 2f(x_0) + f(x_0 - tx)}{t^2}$$

$$f^{(2)+}(x_0;x) = \overline{\lim_{t \downarrow 0}} \frac{f(x_0 + tx) - 2f(x_0) + f(x_0 - tx)}{t^2}.$$

Theorem 2.8. Let x_0 be a minimum point of the function f at the set C and f satisfy the strongly (1,2,1,δ)-Lipschitz condition at the point x_0 with the constant K, $C \subset B(x_0,\delta)$. Then

$$f^{(2)-}(x_0;x) + \lambda d_2^{(2)+}(x_0;x) + \lambda\|x\| d^+(x_0;x) + \lambda\|x\| d^+(x_0;-x) \ge 0$$

for any $\lambda \ge K$ and $x \in X$.

IV. Conclusion

1) In the work, we define $(\alpha, \beta, \nu, \delta, \omega)$- Lipschitz functions at a point in a metric space and study a number of their properties.
2) Using distance functions and $(\alpha, \beta, \nu, \delta, \omega)$- Lipschitz functions at a point, we obtain a number of theorems concerning the exact penalty.
3) Obtain necessary extremum conditions under some constraints in a Banach space.

References

[1] Sadygov M.A. Analysis of nonsmooth optimization problems. Baku, 2002, 125 p.