Control of Loaded Parabolic System with Minimal Energy

Rashad Mammadov[1], Gasim Gasimov[2]
Azerbaijan State Oil Academy, Baku, Azerbaijan
[1]*y_aspirant@yahoo.com*

Abstract— **The problem of control of loaded parabolic system with minimal energy is considered in the work. The statement of the general solution to the considered problem and the statement of the problem in infinite-dimensional phase space are investigated. The solution to the problem on control with minimal energy is given in the work.**

Keywords— minimal energy; loaded parabolic system; general solution; infinite-dimensional phase space; control

I. Introduction

Let the state of the controlled object is described by the function $u(t,x)$, which satisfies the equation

$$u_t = a^2 u_{xk} + \sum_{k=1}^{m} b_k(x) u(t,x_k) + p(t,x) \tag{1}$$

in the oblast $\overline{Q} = \{0 \le x \le 1,\ 0 \le t \le T\}$ and satisfies the initial and boundary conditions

$$u(0,x) = u^0(x), \tag{2}$$

$$u_x(t,0) = 0, \quad u_x(t,1) + \alpha\ u_x(t,1) = 0, \quad \alpha = const > 0 \tag{3}$$

on the border of this oblast. Here $0 < x_1 < x_2 < ... < x_m < 1$ are the known phase points, $b_k(x)$ and $u^0(x)$ are the given functions in $L_2(0,1)$, but $p(t,x)$ is a control parameter.

We will assume that all the functions in $L_2(Q)$ are the admissible controls.

The problem of control with minimal energy consists of determining of such control from the class of admissible controls that, the corresponding solution to the problem satisfies the condition

$$u(T,x) = \psi(x) \tag{4}$$

and the functional

$$I[p] = \|p\|^2_{L_2(Q)} \tag{5}$$

takes on it's minimal admissible value, where $\psi(x) \in L_2(0,1)$ is a given function.

II. Statement of the General Solution to the Problem (1)-(3)

Each concrete admissible control determines the unique solution. This solution can be formulate in the form of Fourier series on nonsingular functions of the boundary problem

$$\begin{aligned} &X''(x) + \lambda^2 X(x) = 0,\ 0 < x < 1; \\ &X'(0) = 0,\ X'(1) + \alpha X(1) = 0. \end{aligned} \tag{6}$$

The system of functions $\{X_n(x)\}$ generates the complete system of orthonormal functions. So using the known method we can find the following expression for the solution of the problem (1)-(3):

$$\begin{aligned} u(x,t) = &\int_0^1 G(x,\xi,t) u^0(\xi) d\xi + \\ &+ \int_0^t \int_0^1 G(x,\xi,t-\tau) \left[\sum_{k=1}^{m} b_k(\xi) u(\tau,x_k) + p(\tau,\xi) \right] d\xi d\tau, \end{aligned} \tag{7}$$

where $G(x,\xi,t)$ is the Grin function and is determined by the formula

$$G(x,\xi,t) = \sum_{n=1}^{\infty} e^{-a^2 \lambda_n^2 t} X_n(x) X_n(\xi),$$

Here λ_n are the nonsingular values of the boundary problem (6).

We'll obtain the system of integral equations corresponding to the functions $u_i(t) = u(t,x_i)$, if consider the values $x = x_i$, $i = 1,2,...,m$ in the equalities (7):

$$u_i(t) = \int_0^t \sum_{k=1}^{m} R_{ik}(t,\tau) u_k(\tau) d\tau + \varphi_i(t),\ i = 1,2,...,m, \tag{8}$$

where

$$R_{ik}(t,\tau) = \int_0^1 G(x_i,\xi,t-\tau) b_k(\xi) d\xi,$$

$$\varphi_i(t) = \int_0^1 G(x_i,\xi,t) u^0(\xi) d\xi + \int_0^t \int_0^1 G(x_i,\xi,t-\tau) f(\tau,\xi) d\xi d\tau.$$

If we denote

$$R(t,\tau)=\{R_{ik}(t,\tau)\}_{\overline{1,m}}^{j=\overline{1,m}},$$

$$\overline{u}(t)=\{u_i(t)\}_{i=1,m},\ \varphi(t)=\{\varphi_i(t)\}_i=\overline{1,m},$$

then we can write the system (8) in the vector form:

$$u(t)=\int_0^t R(t,\tau)u(\tau)d\tau+\varphi(t)$$

and which solution we can organize in the following form:

$$\overline{u}(t)=\sum_{n=0}^{\infty}v_n(t);\ v_0(t)=\varphi(t),$$

$$v_n(t)=\int_0^t R(t,\tau)v_{n-1}(\tau)d\tau,\ n=1,2,...$$

Consequently, the solution to the problem (1)-(3) is determined by the formula (7).

III. Statement of the Problem in Infinite-Dimensional Phase Space

To solve the problem with minimal energy let's formulate it in infinite dimensional phase space. So we'll find the solution to the problem (1)-(3) in the following form:

$$u(t,x)=\sum_{n=1}^{\infty}u_n(t)X_n(x),$$

$$u_n(t)=\int_0^1 u(t,x)X_n(x)dx.$$

Then to determine the coefficients $u_n(t)$ we'll obtain the infinite system of ordinary differential equations

$$\dot{u}_n(t)=-a^2\lambda_n^2u_n(t)+\sum_{n=1}^{\infty}\overline{b}_nu_n(t)+p_n(t) \tag{9}$$

with initial conditions

$$u_n(0)=u_n^0 \tag{10}$$

here $\sum_{k=1}^{m}b_{kn}X_n(x_k)$ a u_n^0, b_{kn}, $P_n(t)$ are the coefficients of Fourier functions $u^0(x)$, $b_k(x)$, $p(t,x)$.

The final condition (4) takes on the following form:

$$u_n(T)=\psi_n \tag{11}$$

where ψ_n are the coefficients of the Fourier functions $\psi(x)$.

In addition to the functions $X_n(x)$, $n=1,2,...$ are generate the complete system of orthonormal functions, the functional takes on the form :

$$I[p]=\int_0^T\sum_{n=1}^{\infty}p_n^2(t)dt \tag{12}$$

The class of admissible controls is the class of infinite dimensional vector functions

$$p(t)=(p_1(t),p_2(t),...),$$

which satisfy the following condition

$$\int_0^T p_n^2(t)dt<\infty .$$

So, it is required to find such admissible control that, corresponding to this control the solution to the problem (9)-(10) satisfies the final condition (11) and in addition to this the functional (12) gets minimal value.

IV. The Solution to the Problem on Control with Minimal Energy

If we'll denote

$$y(t)=(u_1(t),\ u_2(t),...),\ \ p(t)=(p_1(t),\ p_2(t),...),$$
$$y^0=(u_1^0,u_2^0,...),\ \ y^1=(\psi_1,\psi_2,...),$$
$$A\begin{pmatrix}-a^2\lambda_1^2+b_1 & b_2 & ...\\ b_2 & -a^2\lambda_1^2+b_1 & ...\\ ... & ... & ...\end{pmatrix},$$

then we can write the problem (9)-(10) in the vector form:

$$\dot{y}(t)=Ay(t)+P(t), \tag{13}$$

$$y(0)=y^0 \tag{14}$$

But the final condition takes on form

$$y(T)=y^1. \tag{15}$$

We can represent the solution to the problem (13)-(14) by Cauchy formula in the following form:

$$y(t)=e^{At}y^0+\int_0^t e^{A(t-\tau)}p(\tau)d\tau.$$

We'll obtain that the control function which satisfies the torque condition

$$\int_0^T e^{A(T-t)}p(t)dt=c \tag{16},$$

if we consider condition (13), where $c=y^1-e^{AT}y^0$.

We can write the expression (16) in the following form

$$(h_i,p)=c_i,\quad i=1,2.... \tag{17}$$

if we'll denote *i*-th row of the matrix $e^{A(T-t)}$ by $h_i(t)$.

Consequently, it is required to find such control among the admissible controls that, this control will be satisfied the condition (17) and in addition to this its norm will be minimal.

The control $p(t)$ satisfying this condition we'll find in the following form:

$$p(t)=\sum_{i=1}^{\infty}\alpha_ih_i(t). \tag{18}$$

As well as this function satisfies conditions (17), then the vector $\alpha = (\alpha_1, \alpha_2, ...)$ is the solution to the infinite system of equations

$$M\alpha = c \tag{19}$$

where the elements of the matrix M is determined in the following form:

$$M = \{(h_i, h_j)\}i, j = 1.$$

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