Stability Gradient Algorithm on Jordan-Dedekind Structures

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***Abstract*— In this paper a problem of maximization of convex-ordered functions on Jordan-Dedekind structures is considered. In terms of guaranteed estimates it is shown that in problems of optimization of order-convex functions on an Jordan-Dedekind structures the gradient algorithm is stable under small perturbations of the utility function.**

Keywords*— *ordered-convexity; Jordan-Dedekind structures; greedy algorithms; stability

I. Introduction

Introduction initial data of many problem of discrete optimization has the approached character. Therefore the analysis of stability of decisions is actual at fluctuations of parameters of a problem. Numerous publications (see, e.g., [1]) are devoted research of various as pests of stability of scalar and vector problems of discrete optimization. Questions of stability not only decisions of problems of discrete optimization, but also algorithms of their decision (see, e.g., [2, 3]) are actual. One of possible variants of research of stability local (gradient) algorithms is the finding of chance of the guaranteed (relative) estimations at "small" indignations of parameters of a problem. l

In this paper problem of maximization convex-ordered functions on Jordan-Dedekind structures are considering. In terms of guaranteed estimates it is shown that in problems of optimization of order-convex functions on an Jordan-Dedekind structures the gradient algorithm is stable under small perturbations of the utility function. As corollaries we obtain improved guaranteed estimates for accuracy of the gradient algorithm, and also new sufficient conditions for the values of the utility function of the problem under consideration to coincle in the global and gradient extrema.

II. Definitions and Designations

Consider the discrete optimization problem (which we refer to as Problem A)

$$\max\{f(x) \mid x \in H\},$$

where $f(x)$ is a non-decreasing ρ-order-convex function on a partially set H.

Let x^* be an optimal solution of Problem *A*, and let x^g be the point obtained by the following iterative procedure [4]:

III. The Basic Result

Let us remind [4] that a function $f: H \to R$ is called ρ-order-convex on a partially set H if

$$2f(y) - f(x) - f(z) \geq \rho, \forall x \rhd y \rhd z,$$

where ρ is a fixed non-negative number, $x \rhd y$ means that y succeeds x directly in H.

A function $f: H \to R$ is called non-decreasing if $x \prec y$ implies $f(x) \leq f(y)$. The function

$$\Delta^+ f(x) = \max\{f(y) - f(x) \mid x \rhd y,\ x, y \in H\}$$

are called, as usual [1], the right gradient of the function $f(x)$.

The steepness of function $f(x)$ defined as [2]:

$$c(f) = \begin{cases} \min\{(\Delta^+ f(x - \Delta^+ f(y)) / \Delta^+ f(x) : \\ (x,y) \in I\}, if\ I \neq \varnothing, \\ \qquad 0, if\ I = \varnothing, \end{cases}$$

where

$$I = \{(x,y) \,\big|\, \Delta^+ f(x) > \Delta^+ f(y) \geq 0, x \rhd y, \; x, y \in H\}$$

We suppose hereafter that the partially ordered set H satisfies Jordan-Dedekind condition [5]. All maximal chains between comparable elements x and y are of the same length denoted by $h(x,y)$. Besides, we will suppose that the set H has the unique minimal element (zero) denoted by θ.

$$x^{t+1} = \arg\max\{f(y) - f(x^t) : x^t \rhd y, y \in H\},$$
$$t = 0,1,\ldots, x^0 = \theta,$$

which halts on the step τ if either $\Delta^+ f(x^\tau) \leq 0$ or x^τ is the maximal element of the set H (the set H contains the zero θ, as we have stipulated). This point x^g is called the gradient maximum the function $f(x)$ on the set H [4].

By a guaranteed error estimate for the gradient algorithm in Problem *A* we mean a number $\varepsilon \geq 0$

$$\frac{f(x^*)-f(x^g)}{f(x^*)-f(\theta)} \le \varepsilon .$$

By perturbations of problem *A* by means problem *B*

$$\max\{F(x) | x \in H\},$$

where $F(x)$ is a non-decreasing q-order-convex function on a partially set H and $c(F)=c(f)+\delta, \delta \in R, \delta \ge 0$.

Let $\varepsilon(\varepsilon^\delta)$ be a guaranteed error estimate for the gradient algorithm in some unperturbed (perturbed) discrete optimization problem. As usual (see. [3]), we say that the gradient algorithm is stable if $\varepsilon^\delta \le \varepsilon K(\delta)$, where $K(\delta) \to 1$ as $\delta \to 0$.

Theorem. Let ε and ε^δ be guaranteed error estimates for the gradient algorithm in Problems A and B, respectively. Then $\varepsilon^\delta \le \varepsilon$.

To prove Theorem, we need the following lemma.

Lemma. The gradient maximum x^g and the global maximum x^* of any ρ-ordered-convex non-decreasing function $f(x)$ on H are connected by the following relations:

$$\frac{f(x^*)-f(x^g)}{f(x^*)-f(\theta)} \le \left(1-\frac{1}{1+(1-c)(h-1)}\right)^r , \quad (1)$$

where

$$c = c(f), h = max\{h(\theta,x) : x \in H\},$$
$$r = \min\{h(\theta,x) : x \in H^{\max}\},$$

$H^{\max}$ is the set of all maximal elements of the partially ordered set $(H,\prec)$.

Proof of Lemma. By virtue of item of Theorem 4 [4], we have for $y = x^*, x = x^t$

$$f(x^*) \le f(x^t) + h(x^t,x^*)\Delta^+ f(x^t)$$

Together with the fact that

$$\Delta^+ f(x^t) \le (1-c(f))\Delta^+ f(x^{t-1}), t = 1,\dots,r,$$
$$f(x^t) = f(\theta) + \sum_{s=0}^{t-1} \Delta^+ f(x^s), t = 0,\dots,r,$$
$$h(x^t,x^*) \le h,$$

the last inequality yields

$$f(x^*) - f(\theta) \le \sum_{s=0}^{t-1} \Delta^+ f(x^s) + (1-c(f))(h-1)\Delta^+ f(x^t), t = 1,\dots,r .$$

Therefore

$$\sum_{s=0}^{t-1} \beta_s + (1-c)(h-1)\beta_t \ge 1, t = 1,\dots,r,$$

Where

$$\beta_s = \frac{\Delta_s}{\psi}, \psi = f(x^*) - f(\theta), \Delta_s = \Delta^+ f(x^s)$$

Then, by repeating the scheme of the proof of Theorem 4 [4], we obtain estimates (1). Lemma is proved.

Proof of Theorem. According to Lemma

$$\varepsilon = \left(1-\frac{1}{1+(1-c(f))(h-1)}\right)^r , \quad (2)$$

$$\varepsilon^\delta = \left(1-\frac{1}{1+(1-c(F))(h-1)}\right)^r . \quad (3)$$

(2), (3) and the relations bottom follow from theorem

$$1+(1-c(f)-\delta)(h-1) \le 1+(1-c(f))(h-1),$$
$$\frac{1}{1+(1-c(f)-\delta)(h-1)} \ge \frac{1}{1+(1-c(f))(h-1)},$$
$$\varepsilon^\delta = \left(1-\frac{1}{1+(1-c(f)-\delta)(h-1)}\right)^r \le \left(1-\frac{1}{1+(1-c(f))(h-1)}\right)^r = \varepsilon .$$

Theorem is proved.

Corollary. If $c(f) = 1$, then $f(x^*) = f(x^g)$.

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