Linearized and Quadratic Necessary Optimality Conditions in one 2-D Discrete Optimal Control Problem Involving Functional Constraints of Inequality Type

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***Abstract*— The paper is devoted to derivation of linearized and quadratic necessary optimality conditions in one 2-D discrete two-parametric stepwise control problem.**

Keywords*— *2-D discrete systems; functional constraints; admissible process; two-parameter systems

I. Introduction

A step wise problem of optimal control of discrete two-parameter systems is considered. Under the assumption of openness of the control domain, necessary optimality conditions are obtained.

Note that different necessary and sufficient optimality conditions for discrete 2-D control systems were obtained in the papers [1-6] and others.

II. Statement of Problem

Assume that an admissible process is described by the following discrete two-parametric system of equations

$$z_i(t+1,x+1)=$$
$$= f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x)),$$
$$(t,x)\in D_i\,,\; i=\overline{1,3}\,, \tag{1}$$

with boundary conditions

$$\begin{gathered}
z_1(t_0,x)=\alpha_1(x),\quad x=x_0,x_0+1,\dots,X\,,\\
z_1(t,x_0)=\beta_1(t),\quad t=t_0,t_0+1,\dots,t_1\,,\\
z_2(t_1,x)=z_1(t_1,x),\quad x=x_0,x_0+1,\dots,X\,,\\
z_2(t,x_0)=\beta_2(t),\quad t=t_1,t_1+1,\dots,t_2\,,\\
z_3(t_2,x)=z_2(t_2,x),\quad x=x_0,x_0+1,\dots,X\,,\\
z_3(t,x_0)=\beta_3(t),\quad t=t_2,t_2+1,\dots,t_3\,,\\
\alpha_1(x_0)=\beta_1(t_0),\; z_1(t_1,x_0)=\beta_2(t_1),\\
z_2(t_2,x_0)=\beta_3(t_2).
\end{gathered} \tag{2}$$

Here

$$D_i=\{(t,x): t=t_{i-1},t_{i-1}+1,\dots,$$
$$t_i-1;\; x=x_0,x_0+1,\dots,X-1\},\quad i=\overline{1,3},$$

where x_0, X, t_i, $i=\overline{1,3}$ are given, $f_i(t,x,z_i,a_i,b_i,u_i)$, $i=\overline{1,3}$ are the given n-dimensional functions continuous in totality of variables together with partial derivatives with respect to (z_i,a_i,b_i,u_i), $i=\overline{1,3}$ to second order inclusively, $\alpha_1(x)$, $\beta_i(t)$, $i=\overline{1,3}$ are the given n-dimensional discrete vector-functions, $u_i(t,x)$, $i=\overline{1,3}$ are r-dimensional vector-functions of control actions with values from the given non-empty, bounded and convex sets $U_i\subset R^r$, $i=\overline{1,3}$, i.e.

$$u_i(t,x)\in U_i\subset R^r\,,\quad (t,x)\in D_i\,,\quad i=\overline{1,3}\,. \tag{3}$$

The triple $u(t,x)=(u_1(t,x),\,u_2(t,x),\,u_3(t,x))'$ with the above - mentioned properties will be called an admissible control, its appropriate solution $z(t,x)=(z_1(t,x),\,z_2(t,x),\,z_3(t,x))'$ of boundary value problem (1)-(2) an admissible state of the process. Therewith the pair $(u(t,x),\,z(t,x))$ is called an admissible process.

On the solutions of boundary value problem (1)-(2) generated by all possible admissible controls determine the functionals

$$S_j(u)=\sum_{i=1}^{3}\varphi_i^{(j)}(z_i(t_i,X)),\qquad j=\overline{1,p}\,.$$

Here $\varphi_i^{(j)}(z_i)$, $i=\overline{1,3}$, $j=\overline{1,p}$ are the given twice continuously differentiable scalar functions.

The problem consists of minimization of the functional

$$S_0(u)=\sum_{i=1}^{3}\varphi_i^{0}(z_i(t_i,X)), \tag{4}$$

under the constraints

$$S_j(u) \le 0, \quad j = \overline{1,p}. \tag{5}$$

The accessible process satisfying the constraints (5) is called an admissible process.

III. Necessary Optimality Conditions

The admissible process $(u(t,x),\, z(t,x))$ being solution of problem (1)-(4) is called an optimal process.

Let $(u(t,x),\, z(t,x))$ be a fixed admissible process. Further we'll use the following denotation:

$$H_i^{(j)}(t,x,z_i,a_i,b_i,u_i,\psi_i) = \psi_i^{(j)'} f_i(t,x,z_i,a_i,b_i,u_i),$$

$$\frac{\partial f_i[t,x]}{\partial u_i} \equiv \frac{\partial f_i(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x))}{\partial u_i},$$

$$\frac{\partial^2 H_i^{(j)}[t,x]}{\partial z_i^2} \equiv$$

$$= \frac{\partial^2 H_i^{(j)}(t,x,z_i(t,x),z_i(t+1,x),z_i(t,x+1),u_i(t,x),\psi_i(t,x))}{\partial z_i^2},$$

where $\psi_i^{(j)} = \psi_i^{(j)}(t,x)$, $i = \overline{1,3}$ are n- dimensional vector-functions of conjugated variables being the solutions of the problem

$$\psi_i^{(j)}(t-1,x-1) = \frac{\partial H_i^{(j)}[t,x]}{\partial z_i} + \frac{\partial H_i^{(j)}[t-1,x]}{\partial a_i} + $$
$$+ \frac{\partial H_i^{(j)}[t,x-1]}{\partial b_i}, \; i = \overline{1,3}, \tag{5}$$

$$\psi_1^{(j)}(t_1-1,X-1) = \psi_2^{(j)}(t_1-1,X-1) - $$
$$- \frac{\partial \varphi_1^{(j)}(z_1(t_1,X))}{\partial z_1},$$

$$\psi_1^{(j)}(t-1,X-1) = \frac{\partial H_1^{(j)}[t-1,X-1]}{\partial b_1},$$

$$\psi_1^{(j)}(t_1-1,x-1) = \psi_2^{(j)}(t_1-1,x-1) + $$
$$+ \frac{\partial H_1^{(j)}[t_1-1,x]}{\partial a_1} - \frac{\partial H_2^{(j)}[t_1-1,x]}{\partial a_2},$$

$$\psi_2^{(j)}(t_2-1,X-1) = \psi_3^{(j)}(t_2-1,X-1) - $$
$$- \frac{\partial \varphi_2^{(j)}(z_2(t_2,X))}{\partial z_2},$$

$$\psi_2^{(j)}(t_2-1,x-1) = \psi_3^{(j)}(t_2-1,x-1) + $$
$$+ \frac{\partial H_2^{(j)}[t_2-1,x]}{\partial a_2} - \frac{\partial H_3^{(j)}[t_2-1,x]}{\partial a_3},$$

$$\psi_2^{(j)}(t-1,X-1) = \frac{\partial H_2^{(j)}[t,X-1]}{\partial b_2},$$

$$\psi_3^{(j)}(t_3-1,X-1) = -\frac{\partial \varphi_3^{(j)}(z_3(t_3,X))}{\partial z_3},$$

$$\psi_3^{(j)}(t_3-1,x-1) = \frac{\partial H_3^{(j)}[t_3-1,x]}{\partial a_3},$$

$$\psi_3^{(j)}(t-1,X-1) = \frac{\partial H_3^{(j)}[t,X-1]}{\partial b_3}. \tag{6}$$

Let by definition

$$I(u) = \{j;\, S_j(u) = 0,\; j = \overline{1,p}\},$$
$$J(u) = \{0\} \cup I(u).$$

Theorem 1. Along the optimal process $(u(t,x),\, z(t,x))$

$$\min_{j \in J(u)} \sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \frac{\partial H_i^{(j)}[t,x]}{\partial u_i} (v_i(t,x) - u_i(t,x)) \le 0$$

for all $v_i(t,x) \in U_i$, $(t,x) \in D_i$, $i = \overline{1,3}$. (7)

Relation (7), being a necessary optimality condition of first order, is an analogy of the linearized maximum condition for problem (1)-(2).

Definition 1. The admissible control $v(t,x) = (v_1(t,x),\, v_2(t,x),\, v_3(t,x))'$ is called critical with respect to the extremum $u(t,x) = (u_1(t,x),\, u_2(t,x),\, u_3(t,x))$, if

$$\min_{j \in J(u)} \sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \frac{\partial H_i^{(j)}[t,x]}{\partial u_i} (v_i(t,x) - u_i(t,x)) \equiv 0,$$
$$i = \overline{1,3}. \tag{8}$$

The extremum $u(t,x)$ is said to be quasi-singular in problem (1)-(5) if there exists the appropriate critical admissible control $v(t,x) \ne u(t,x)$.

Now, by $J_1(u)$ denote a maximal subset of indices from $J(u)$ such that

$$\sum_{t=t_{i-1}}^{t_i-1} \sum_{x=x_0}^{X-1} \frac{\partial H_i^{(j)}[t,x]}{\partial u_i} = 0, \quad j \in J_1(u), \quad i = \overline{1,3}$$

for all admissible controls $v(t,x)$ critical with respect to $u(t,x)$.

Assume that in system (2.1)

$$f_i(t,x,z_i,a_i,b_i,u_i) = B_i(t,x)b_i + F(t,x,z_i,a_i,u_i). \tag{9}$$

Further, the necessary optimality condition of quasi-singular controls is proved in problem (1)-(5), (9).

IV. Conclusion

Linearized and quadratic necessary optimality conditions are established for a stepwise problem of optimal control of discrete two-parameter systems involving functional constraints.

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